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ScienceDirect
Journal of Algebra 319 (2008) 2626–2651
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Complete cohomology and Gorensteinness of schemes ☆

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Received 20 April 2007

Available online 21 December 2007

Communicated by Steven Dale Cutkosky

Abstract

We develop and study Tate and complete cohomology theory in the category of sheaves of $\mathcal{O}_X$-modules. Different approaches are included. We study the properties of these theories and show their power in reflecting the Gorensteinness of the underlying scheme. The connection of these two theories will be discussed.

Keywords: Gorenstein scheme; Locally free sheaf; Cohomology of sheaves; Homological dimension; Complete cohomology

Contents

☆ This research was in part supported by a grant from IPM (Nos. 86130019 and 86130024). The first author thanks the Center of Excellence of Algebraic Methods and Applications of Isfahan University of Technology (IUT-CEAMA). The second and the third authors thank the Center of Excellence for Mathematics (University of Isfahan).

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.11.020

1. Introduction

In this paper we aim to verify if and how Tate and complete cohomology reflect or detect the Gorensteinness of the underlying schemes. Tate cohomology goes back to the Tate's observation [T] that the homology and cohomology groups of G, when G is a finite group, can be fit together to form a single cohomology theory. This idea was generalized by Farrell to any group G of finite virtual cohomological dimension and by Buchweitz to two sided noetherian Gorenstein rings [Bu]. Recent expositions of Tate cohomology over noetherian rings can be found in [AM] and [AS]. On the other hand, as an attempt for generalizing the theory to arbitrary groups, complete cohomology was developed independently by Vogel [Goi] and Mislin [Mis]. Benson and Carlson [BC] also constructed a theory isomorphic to that of Vogel and that of Mislin. The alternative approach to the theory via injectives and a comparison of two theories is considered in [N].

The power of these theories in reflecting the Gorensteinness of the underlying ring, in module categories over commutative noetherian rings, is already established, see, e.g. [AV, Section 6], where they show that $(R, \mathfrak{m}, k)$ is Gorenstein (respectively, regular) if complete cohomology of k is finitely generated (respectively, zero) for one single integer n. Several authors also have studied these theories in different abelian categories. For example, Krause [K] develop a Tate cohomology over a separated noetherian scheme, using the left adjoint functor of the inclusion functor i from the homotopy category of totally acyclic complexes of injectives to the homotopy category of injectives, see [K, Lemma 7.3].

Our approach in this paper, is based on the explicit description of these theories over schemes. We show that these theories can be defined and, as their prototypes in module theories, can describe some properties of schemes. Tate cohomology sheaves, $\widehat{\mathcal{E}xt}$, are defined using the notion of totally reflexive sheave, which is an analogue of a module in Auslander's G-class [ABr]. These sheaves are, in fact, syzygies of acyclic complexes of locally free sheaves of finite ranks, which remain exact under dualization. These complexes will be called totally acyclic. Locally free sheaves of finite rank are totally reflexive, as well as sheaves associated to modules in Auslander's G-class over a noetherian affine scheme $X = \operatorname{Spec}(R)$. Over locally Gorenstein schemes, schemes whose their local rings are all Gorenstein rings, totally reflexive sheaves are locally Cohen–Macaulay ones. Furthermore, over projective schemes these sheaves are syzygies of a $\mathcal{H}om(\ , \mathcal{O}_X)$-exact exact sequence of dissocié sheaves. Now, let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that one of its syzygies, say Ω, is totally reflexive. Let $\mathcal{T}^\bullet$ be the totally acyclic complex related to Ω. Then, for each $n \in \mathbb{Z}$ and each $\mathcal{O}_X$-module $\mathcal{F}'$ that admits a locally free resolution, a Tate cohomology sheaf is defined by the equality $\widehat{\mathcal{E}xt}^n(\mathcal{F},\mathcal{F}') = \mathcal{H}_{-n}\mathcal{H}om(\mathcal{T}^\bullet,\mathcal{F}')$. We show that this definition is independent of the choice of $\mathcal{T}^\bullet$ and that this theory shares basic properties with ordinary cohomology theory.

Complete cohomology, assigns to each pair $(\mathcal{F},\mathcal{G})$ of $\mathcal{O}_X$-modules sheaves $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G})$, $n \in \mathbb{Z}$, see Section 4.2 below. This assignment gives us a bifunctor from $\mathfrak{Mod}(X)$ to itself which is covariant in $\mathcal{F}$ and contravariant in $\mathcal{G}$. It is shown that vanishing of these cohomology sheaves characterize sheaves of finite locally injective dimension: an $\mathcal{O}_X$-module $\mathcal{G}$ is of finite locally injective dimension, denoted $\operatorname{lid}\mathcal{G} < \infty$, if there exists a cover of X by open sets U such that $\mathcal{G}|_U$ is of finite injective dimension as $\mathcal{O}_X|_U$-module. We show that an $\mathcal{O}_X$-module $\mathcal{G}$ is of finite locally injective dimension if and only if $\widetilde{\mathcal{E}\mathrm{xt}}^0(\mathcal{G},\mathcal{G}) = 0$. This provides a cohomological criteria for the Gorensteinness of the locally noetherian schemes: a locally noetherian scheme X of finite dimension is locally Gorenstein if and only if $\widetilde{\mathcal{E}\mathrm{xt}}^0_{\mathfrak{Q}(X)}(\mathcal{O}_X,\mathcal{O}_X) = 0$, which is equivalent to say that $\widetilde{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\ ,\mathcal{O}_X) = 0$, for all integers i. The subscript $\mathfrak{Q}(X)$ emphasize that Ext sheaves are computed in $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules. As another criteria, we show that locally noetherian scheme X of dimension d is locally Gorenstein if for all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ and for a single integer n, the cohomology sheaf $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G})$ is coherent. These results show that complete cohomology theory is a powerful tool in study the property of the underlying schemes.

2. Totally reflexive sheaves

We begin this section by recalling the notion of totally reflexive modules, that are prototype for the definition of totally reflexive sheaves. Let A be a commutative noetherian ring. An exact sequence

$$T^\bullet: \quad \cdots \to T_{n+1} \to T_n \to T_{n-1} \to \cdots$$

of finitely generated projective A-modules is called totally acyclic if the dual complex $\operatorname{Hom}_A(T^\bullet, A)$ is exact. A finitely generated A-module G is called totally reflexive if it is isomorphic to a kernel of a totally acyclic complex. It is easy to check that G is totally reflexive if it is reflexive and, in addition,

$$\operatorname{Ext}^n_A(G,A) = 0 = \operatorname{Ext}^n_A\bigl(\operatorname{Hom}_A(G,A),A\bigr), \quad \text{for all } n > 0.$$

The class of totally reflexive A-modules will be denoted by $\mathrm{G}(A)$. Using the modules in $\mathrm{G}(A)$, Auslander introduced an invariant for any finitely generated A-module M, called Gorenstein dimension (G-dimension for short), denoted $\operatorname{G-dim}_A M$. It is finite for all finite modules over a Gorenstein ring [Got]. Moreover, it is finer than projective dimension and we have equality when projective dimension is finite. For more details see [ABr].

Definition 2.1. Let $(X,\mathcal{O}_X)$ be a ringed space. An exact complex $\mathcal{L}^\bullet$,

$$\mathcal{L}^\bullet: \quad \cdots \to \mathcal{L}_{n+1} \to \mathcal{L}_n \to \mathcal{L}_{n-1} \to \cdots$$

of locally free $\mathcal{O}_X$-modules of finite ranks is called *totally acyclic* if for any locally free $\mathcal{O}_X$-module $\mathcal{L}'$ the complex $\mathcal{H}om(\mathcal{L}^\bullet,\mathcal{L}')$ is also exact.

We say that $\mathcal{O}_X$-module $\mathcal{F}$ is *totally reflexive* if it is isomorphic to a kernel of a totally acyclic complex.

For any ringed space X the class of totally reflexive $\mathcal{O}_X$-modules will be denoted by $\mathrm{G}(X)$. It is clear that $\mathrm{G}(X)$ contains all locally free sheaves of finite rank.

2.1. Examples and descriptions

In this subsection, we give some examples of totally reflexive sheaves and present some descriptions for these sheaves. We begin by a lemma.

Lemma 2.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let*

$$\mathcal{L}^\bullet\colon\quad \cdots \to \mathcal{L}_{n+1} \to \mathcal{L}_n \to \mathcal{L}_{n-1} \to \cdots$$

be an exact complex of locally free $\mathcal{O}_X$-modules of finite rank. Assume that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{O}_X)$ is exact. Then, for any locally free $\mathcal{O}_X$-module $\mathcal{L}'$ the complex $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{L}')$ is also exact.

Proof. Let $\mathcal{L}'$ be a locally free sheaf of finite rank. Let x be an arbitrary element of X. So there exists a neighborhood V of x such that $\mathcal{L}'|_V$ is a free $\mathcal{O}_X|_V$-module. So the complex $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{L}')|_V$ is exact. This implies that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}^\bullet, \mathcal{L}')_x$ is exact. Since x was arbitrary, the result follows. $\square$

Example 2.1.2. Over noetherian affine schemes, totally reflexive sheaves correspond exactly to totally reflexive modules. This follows from the fact that for any noetherian ring A and any A-module N, the map $N \to \widetilde{N}$ gives an exact functor from the category of A-modules to the category of $\mathcal{O}_X$-modules that commutes with arbitrary direct sums, where $\widetilde{N}$ denote the sheaf associated to N on $X = \operatorname{Spec} A$ (see [H, II.5.2]).

Example 2.1.3. Let G be a finite cyclic group of order n with generator t. Let N denote the norm element $1 + t + t^2 + \cdots + t^{n-1}$ of $\mathbb{Z}G$. It is known that there is a periodic resolution of period 2

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z}G \longrightarrow \cdots,$$

where $\mathbb{Z} = \operatorname{Ker}(\mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G)$, $\alpha(g) = (t-1)g$ and $\epsilon(g) = Ng$, for all $g \in \mathbb{Z}G$, see e.g. [Br, I.6].

Now, let X be a topological space. Give G and $\mathbb{Z}G$ the discrete topology and for any open set $U \subseteq X$, let $\mathcal{O}_X(U)$ be the ring of continuous maps of U into $\mathbb{Z}G$. Then clearly $\mathcal{O}_X$, with the usual restriction map, is a sheaf of rings. Now let $\mathcal{F}$ be the constant sheaf on X determined by $\mathbb{Z}$. Hence with the trivial G-action, $\mathcal{F}$ is an $\mathcal{O}_X$-module. We show that it is totally reflexive. The maps in the above periodic resolution induce morphisms of sheaves $\mathcal{O}_X \xrightarrow{k} \mathcal{O}_X$ and $\mathcal{O}_X \xrightarrow{l} \mathcal{O}_X$, where for any open set U of X, $k(U)\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is defined by $k(U)(f) = \epsilon \circ f$ and $l(U)\colon \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is defined by $l(U)(f) = \alpha \circ f$.

As a result we get an exact sequence

$$\mathcal{O}^\bullet\colon\quad \cdots \longrightarrow \mathcal{O}_X \xrightarrow{l} \mathcal{O}_X \xrightarrow{k} \mathcal{O}_X \xrightarrow{l} \mathcal{O}_X \xrightarrow{k} \mathcal{O}_X \longrightarrow \cdots$$

of sheaves of $\mathcal{O}_X$-modules. The exactness of the dual complex $\mathcal{O}^{\bullet\vee} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}^\bullet, \mathcal{O}_X)$ follows from the fact that $\mathcal{O}^{\bullet\vee}$ is isomorphic to $\mathcal{O}^\bullet$ shifted one degree. Hence $\mathcal{O}^\bullet$ is a totally acyclic complex. Now, it is not difficult to see that $\mathcal{F} \cong \operatorname{Ker}(\mathcal{O}_X \xrightarrow{k} \mathcal{O}_X)$. Hence $\mathcal{F}$ is a totally reflexive $\mathcal{O}_X$-module.

Example 2.1.4. Let S be a graded noetherian ring which is finitely generated by S_1 as an S_0-algebra. Let M be a graded totally reflexive S-module, then one can check that $\widetilde{M}$ is totally reflexive.

Remark 2.1.5. The notion of a totally reflexive sheaf over a scheme can be defined locally by saying if a scheme has a covering by open affines on which Definition 2.1 is satisfied. Then to be sure we have got the right notion, one should prove that it is equivalent to say 'there exists an open affine cover' or 'for every open affine cover.' Of course this can be done, but for our purposes in Section 3, this definition does not work. So we prefer to work with the above definition.

Proposition 2.1.6. *Let X be a noetherian scheme and let $\mathcal{L}^\bullet$ be a complex of coherent $\mathcal{O}_X$-modules. Assume that for all $x \in X$, $\mathcal{L}^\bullet_x$ is a totally acyclic complex of $\mathcal{O}_x$-modules. Then $\mathcal{L}^\bullet$ is totally acyclic.*

Proof. Since for any integer i, $(\mathcal{L}^\bullet_x)_i$ is free of finite rank, we may deduce that each $\mathcal{L}_i$ is locally free of finite rank. The exactness of the complex $\mathcal{L}^\bullet$ and its dual $\mathcal{H}om(\mathcal{L}^\bullet, \mathcal{O}_X)$, follows from the exactness of their stalks. So by Lemma 2.1.1, we are done. □

As in the module case, totally reflexive sheaves can be characterized in term of the vanishing of '$\mathcal{E}xt$' sheaves. We leave the easy proof of the following proposition, to the reader. Throughout, $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ denotes the dual of $\mathcal{F}$.

Proposition 2.1.7. *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ belongs to* $\mathrm{G}(X)$ *if and only if it satisfies the following conditions:*

(i) *$\mathcal{F}$ is reflexive.*
(ii) *$\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0 = \mathcal{E}xt^i(\mathcal{F}^\vee, \mathcal{O}_X)$, for all $i > 0$.*
(iii) *$\mathcal{F}$ and $\mathcal{F}^\vee$ have resolutions by locally free $\mathcal{O}_X$-modules of finite rank.*

The following corollary follows from the above two propositions.

Corollary 2.1.8. *Let X be a noetherian scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module such that both $\mathcal{F}$ and $\mathcal{F}^\vee$ admit locally free resolutions of finite ranks. Then $\mathcal{F}$ is in* $\mathrm{G}(X)$ *if and only if for all $x \in X$, $\mathcal{F}_x$ is a totally reflexive $\mathcal{O}_x$-module.*

Remark 2.1.9. Let X be a projective scheme over a noetherian ring A. Then any totally reflexive sheaf $\mathcal{G}$ is a syzygy of a $\mathcal{H}om(\ ,\mathcal{O}_X)$-exact exact sequence of dissocié sheaves. To see this, by [H, II.5.18] $\mathcal{G}$ and $\mathcal{G}^\vee$ admit resolutions $\mathcal{L}^\bullet$ and $\mathcal{L}'^\bullet$ by dissocié sheaves, respectively. Then paste resolutions $\mathcal{L}^\bullet$ and $\mathcal{L}'^{\bullet\vee}$ together. The exactness of the resulting complex and its dual follows from Proposition 2.1.7.

Let X be a noetherian scheme and let $\mathfrak{Coh}(X)$ denote the category of coherent sheaves of $\mathcal{O}_X$-modules. We say that $\mathfrak{Coh}(X)$ has enough locally free sheaves if every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank). For instance, it is proved by Kleiman that this is the case when X is a noetherian, integral, separated, locally factorial scheme (see [Bo]).

Example 2.1.10. In this example we show that if $\mathfrak{Coh}(X)$ has enough locally free sheaves, where X is a locally Gorenstein scheme, then totally reflexive sheaves are precisely locally Cohen–Macaulay ones. To see this, let $\mathcal{F}$ be a locally CM sheaf on such a scheme. Let $x \in X$ be such that $\mathcal{F}_x \neq 0$. Since $\mathcal{O}_x$ is Gorenstein, $\operatorname{G-dim}_{\mathcal{O}_x} \mathcal{F}_x$ is finite. So it satisfies the Auslander–Bridger formula, that is

$$\operatorname{G-dim}_{\mathcal{O}_x} \mathcal{F}_x = \operatorname{depth} \mathcal{O}_x - \operatorname{depth} \mathcal{F}_x.$$

But since $\mathcal{F}$ is locally CM, $\operatorname{depth} \mathcal{F}_x = \dim \mathcal{O}_x$. Hence $\operatorname{G-dim}_{\mathcal{O}_x} \mathcal{F}_x = 0$, for all $x \in X$. So, by the above corollary, $\mathcal{F} \in \mathrm{G}(X)$.

2.1.11. We recall that a class $\mathcal{X}$ of objects of a category $\mathcal{C}$ is projectively resolving if $\mathcal{X}$ is closed under extensions and under kernel of epimorphisms. Assume that X is a noetherian scheme such that $\mathfrak{Coh}(X)$ has enough locally free sheaves. Then the subcategory $\mathrm{G}(X)$ is projectively resolving. To see this, let $0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{G} \to 0$ be an exact sequence of coherent sheaves such that $\mathcal{G} \in \mathrm{G}(X)$. We assume that one of $\mathcal{K}$ or $\mathcal{F}$ is in $\mathrm{G}(X)$ and show that the other also is in $\mathrm{G}(X)$. Since $\mathcal{E}xt^1(\mathcal{G}, \mathcal{O}_X) = 0$, the sequence $0 \to \mathcal{G}^\vee \to \mathcal{F}^\vee \to \mathcal{K}^\vee \to 0$ is exact. Applying the dual functor one time more, gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d, "\mu^{\mathcal{K}}"] & \mathcal{F} \arrow[r] \arrow[d, "\mu^{\mathcal{F}}"] & \mathcal{G} \arrow[r] \arrow[d, "\mu^{\mathcal{G}}", "\cong"'] & 0 \\
0 \arrow[r] & \mathcal{K}^{\vee\vee} \arrow[r] & \mathcal{F}^{\vee\vee} \arrow[r] & \mathcal{G}^{\vee\vee}. &
\end{tikzcd}$$

By assumption $\mu^{\mathcal{G}}$ and one of $\mu^{\mathcal{K}}$ or $\mu^{\mathcal{F}}$ are isomorphisms. Hence for any $x \in X$, $(\mu^{\mathcal{G}})_x$ and one of $(\mu^{\mathcal{K}})_x$ or $(\mu^{\mathcal{F}})_x$ are isomorphisms. Five lemma implies that the other one is isomorphism, for any $x \in X$. Hence the map itself is isomorphism. The vanishing of $\mathcal{E}xt$ follows easily from the long exact sequence of cohomology sheaves. Hence in view of Proposition 2.1.7 we are done.

2.1.12. Let X be a locally Gorenstein scheme of finite dimension. Then any exact sequence

$$\mathcal{L}^\bullet: \quad \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{L}_{-1} \to \cdots$$

of locally free $\mathcal{O}_X$-modules of finite rank is totally acyclic. To see this, in view of Lemma 2.1.1, we just need to prove that $\mathcal{L}^\bullet$ is $\mathcal{H}om(\ ,\mathcal{O}_X)$-exact. To this end, we show that the stalks of the complex $\mathcal{H}om(\mathcal{L}^\bullet, \mathcal{O}_X)$, at any $x \in X$, are exact. So let $x \in X$ and consider the complex

$$\cdots \to \mathcal{H}om(\mathcal{L}_{-1}, \mathcal{O}_X)_x \to \mathcal{H}om(\mathcal{L}_0, \mathcal{O}_X)_x \to \mathcal{H}om(\mathcal{L}_1, \mathcal{O}_X)_x \to \cdots.$$

Since $\mathcal{L}_i$ for any integer i is of finite rank, this complex is isomorphic to the complex

$$\cdots \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{L}_{-1x}, \mathcal{O}_x) \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{L}_{0x}, \mathcal{O}_x) \to \operatorname{Hom}_{\mathcal{O}_x}(\mathcal{L}_{1x}, \mathcal{O}_x) \to \cdots$$

of $\mathcal{O}_x$-modules. But since $\mathcal{O}_x$ is a Gorenstein local ring, its injective dimension is finite. This implies that the localized complex and hence the complex $\mathcal{H}om(\mathcal{L}^\bullet, \mathcal{O}_X)_x$ is exact.

Remark 2.1.13. It is known that on a normal variety, locally free sheaves of rank 1 correspond to Cartier divisors and reflexive sheaves of rank 1 correspond to Weil divisors. A question that is posed by David A. Cox is that: Where do totally reflexive sheaves of rank 1 fit into this picture?

Remark 2.1.14. Theory of sheaves of modules over schemes has many features in common with the topological bundles. In this remark, we use this correspondence to study $\mathrm{G}(X)$, for the ringed space $(X, \mathcal{O}_{top})$, when $sp(X)$ is a compact space.

Recall that a vector bundle over a ringed space X is a locally free $\mathcal{O}_X$-module of finite rank. Now fix a topological space X and consider the ringed space $(X, \mathcal{O}_{top})$, where $\mathcal{O}_{top}$ is the sheaf of real valued continuous functions on X. It is known that there is an equivalence between the category of vector bundles over the ringed space $(X, \mathcal{O}_{top})$, i.e. locally free $\mathcal{O}_{top}$-modules, and the category $\mathrm{VB}(X)$ of real topological vector bundles over X. Now if one consider an exact sequence of vector bundles over a space X, since the kernel of morphisms between vector bundles are also vector bundles, all its syzygies are again vector bundles over X. This fact, in view of the above mentioned equivalence of categories, implies that $\mathrm{G}(X)$ is equal to the set of all locally free sheaves of finite ranks.

On the other hand, if X is compact, Swan's theorem establish an equivalence of categories between the category of vector bundles over a space X and the category of finitely generated projective modules over $C(X)$, the ring of continuous functions on X. So every finitely generated Gorenstein projective module over $C(X)$ is projective.

2.2. *Gorenstein homological dimension*

Let X be a noetherian scheme and $\mathcal{F}$ be a coherent sheaf. Assume that $\mathcal{F}$ has a locally free resolution of finite rank. In this case, one can define the homological dimension of $\mathcal{F}$, denoted $\mathrm{hd}(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ (or $+\infty$ if there is no finite one). Using the notion of totally reflexive sheaves, we are able to introduce a refinement of this invariant.

Definition 2.2.1. Let $(X, \mathcal{O}_X)$ be a noetherian scheme and $\mathcal{F} \neq 0$ be an $\mathcal{O}_X$-module admitting a locally free resolution of finite rank. We say that Gorenstein homological dimension of $\mathcal{F}$ is zero, denoted $\mathrm{Ghd}\,\mathcal{F} = 0$, if $\mathcal{F} \in \mathrm{G}(X)$. We set $\mathrm{Ghd}\,\mathcal{F} = n$, if n is the smallest integer such that the nth syzygy of a locally free resolution of $\mathcal{F}$ belongs to $\mathrm{G}(X)$. We set $\mathrm{Ghd}\,\mathcal{F} = \infty$, if such a resolution does not exist. In the case $\mathcal{F} = 0$, we set $\mathrm{Ghd}\,\mathcal{F} = -\infty$.

Proposition 2.2.2. *Assume that X is a noetherian scheme of finite dimension n and $\mathcal{F}$ is an $\mathcal{O}_X$-module admitting a locally free resolution of finite rank. Then there is an inequality* $\mathrm{Ghd}\,\mathcal{F} \leqslant \mathrm{hd}\,\mathcal{F}$*, with equality when* $\mathrm{hd}\,\mathcal{F} < \infty$.

Proof. It is clear that $\mathrm{Ghd}\,\mathcal{F} \leqslant \mathrm{hd}\,\mathcal{F}$. Assume that $\mathrm{hd}\,\mathcal{F} < \infty$. To establish the equality, we use induction on $\mathrm{Ghd}\,\mathcal{F}$. Let $\mathrm{Ghd}\,\mathcal{F} = 0$. Consider the short exact sequence $0 \to \mathcal{F} \to \mathcal{L}^0 \to \mathcal{K}^0 \to 0$, in which $\mathcal{L}^0$ is locally free and $\mathrm{Ghd}\,\mathcal{K}^0 = 0$. The long exact sequence of '$\mathcal{E}xt$' sheaves arising from this short exact sequence implies that $\mathcal{E}xt^j(\mathcal{F}, \mathcal{G}) \cong \mathcal{E}xt^{j+1}(\mathcal{K}^0, \mathcal{G})$, for all $\mathcal{G} \in \mathfrak{Mod}(X)$ and all $j \in \mathbb{N}$. On the other hand, by the same short exact sequence

$\operatorname{hd}\mathcal{K}^0 < \infty$, which in view of the fact that $\dim X = n$, we may deduce that $\operatorname{hd}\mathcal{K}^0 \leqslant n$. Consider now the exact sequences $0 \to \mathcal{K}^i \to \mathcal{L}^{i+1} \to \mathcal{K}^{i+1} \to 0$, $i \geqslant 0$, where $\mathcal{L}^{i+1}$ is locally free and $\operatorname{hd}\mathcal{K}^{i+1} \leqslant n$, and apply the functor $\mathcal{H}om(\ ,\mathcal{G})$ on them to get

$$\mathcal{E}xt^j(\mathcal{F},\mathcal{G}) \cong \mathcal{E}xt^{n+j}\big(\mathcal{K}^{n-1},\mathcal{G}\big).$$

But since $\operatorname{hd}\mathcal{K}^{n-1} \leqslant n$, we deduce from the above isomorphisms that $\mathcal{E}xt^j(\mathcal{F},\mathcal{G}) = 0$, for all $j > 0$ and all $\mathcal{G} \in \mathfrak{Mod}(X)$. This implies that $\operatorname{hd}\mathcal{F} = 0$. Now suppose, inductively, that $\operatorname{Ghd}\mathcal{F} = g > 1$. Consider the short exact sequence $0 \to \mathcal{K} \to \mathcal{L} \to \mathcal{F} \to 0$, in which $\mathcal{L}$ is locally free and $\operatorname{Ghd}\mathcal{K} = g - 1$. By induction assumption, we get $\operatorname{hd}\mathcal{K} = g - 1$. Hence $\operatorname{hd}\mathcal{F} \leqslant g$. So we have equality. □

Theorem 2.2.3. *Let X be a locally Gorenstein scheme of finite dimension. Assume that $\mathfrak{Coh}(X)$ has enough locally frees. Then any coherent sheaf $\mathcal{F}$ is of finite Gorenstein homological dimension.*

Proof. Let $\dim X = n$. Since X is locally Gorenstein, $\mathcal{O}_x$, for any $x \in X$, is a Gorenstein local ring and so by [ABr, 4.20], Gorenstein dimension of any module over such ring is finite. Hence if $\mathcal{F}$ is a coherent sheaf, for any $x \in X$, we have $\text{G-dim}_{\mathcal{O}_x} \mathcal{F}_x \leqslant \dim \mathcal{O}_x \leqslant \dim X$. So all stalks of the nth syzygy of $\mathcal{F}$ are in Auslander's G-class. This, in view of Corollary 2.1.8, implies that the nth syzygy itself is in $\mathrm{G}(X)$ and so $\operatorname{Ghd}\mathcal{F} < \infty$. □

One can easily check that an affine scheme X is locally Gorenstein if and only if for any coherent sheaf $\mathcal{F}$, $\operatorname{Ghd}\mathcal{F} < \infty$. This can be seen, using the faithfully exact functor $\sim$.

Proposition 2.2.4. *Let $X = \operatorname{Proj} S$, where S is a graded ring which is finitely generated by S_1 as an S_0-algebra. Let $\mathcal{F}$ be a coherent sheaf with $\operatorname{Ghd}\mathcal{F} = g < \infty$. Then $\operatorname{Ghd}\mathcal{F}(n) \leqslant g$, for any $n \in \mathbb{N}$.*

Proof. We argue by induction on g. Assume that $g = 0$. It is plain that $\mathcal{F}(n)$ is reflexive and both $\mathcal{F}(n)$ and $\mathcal{F}(n)^\vee$ have resolutions by locally free sheaves of finite ranks, in fact, by dissocié sheaves. Let $\mathcal{L}^\bullet \to \mathcal{F}$ be a locally free resolution of $\mathcal{F}$. Then $\mathcal{L}^\bullet(n) \to \mathcal{F}(n)$ is a locally free resolution of $\mathcal{F}(n)$ and we have $\mathcal{H}om(\mathcal{L}^\bullet(n), \mathcal{O}_X) \cong \mathcal{H}om(\mathcal{L}^\bullet, \mathcal{O}_X) \otimes \mathcal{O}_X(-n)$. But since $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) = 0$, for all $i > 0$ and $\mathcal{O}_X(-n)$ is flat on X, we may deduce that $\mathcal{E}xt^i(\mathcal{F}(n), \mathcal{O}_X) = 0$, for all $i > 0$. Similarly, we can show that $\mathcal{E}xt^i(\mathcal{F}(n)^\vee, \mathcal{O}_X) = 0$, for all $i > 0$. Hence by 2.1.7, $\operatorname{Ghd}\mathcal{F}(n) = 0$. The inductive step, now can be completed easily. □

We recall that a closed subscheme X of $\mathbb{P}^n_k$, k an algebraically closed field, is called arithmetically Gorenstein (AG) if its homogeneous coordinate ring $S(X) = k[x_0, \ldots, x_n]/I_X$ is a Gorenstein ring.

Theorem 2.2.5. *Let X be an AG scheme with $\dim S(X) = d$. Then any coherent $\mathcal{O}_X$-module $\mathcal{F}$ is of finite Gorenstein homological dimension.*

Proof. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let M be the finitely generated submodule of $\Gamma_*\mathcal{F}$ generated by the finite number of global sections in $\Gamma(X, \mathcal{F}(n))$ that generate $\mathcal{F}(n)$, see [H, II.5.17].

Then $\mathcal{F} \cong \widetilde{M}$. Since $S = S(X)$ is Gorenstein of dimension d, $\text{G-dim}_S M \leqslant d$. So there is an exact sequence

$$0 \to L \to \bigoplus_n S(n)^{\beta_{dn}} \to \cdots \to \bigoplus_n S(n)^{\beta_{0n}} \to M \to 0,$$

in which L is in Auslander's G-class. Hence, by Example 2.1.4, the sheaf associated to L, $\widetilde{L}$, is totally reflexive. This implies the result. $\square$

Let X be a noetherian scheme and $\mathfrak{Coh}(X)$ has enough locally frees. Let $\mathcal{F}$ be a coherent sheaf of finite Gorenstein homological dimension. It is easy to see that in this situation, $\operatorname{Ghd} \mathcal{F} = \sup_{x \in X} \text{G-dim}_{\mathcal{O}_x} \mathcal{F}_x$.

2.2.6. Let $\operatorname{Ghd} \mathcal{F} < \infty$. Then

$$\begin{aligned} \operatorname{Ghd} \mathcal{F} &= \sup\bigl\{i \in \mathbb{N}_0 \bigm| \mathcal{E}xt^i(\mathcal{F}, \mathcal{L}) \neq 0 \text{ for some locally free sheaf } \mathcal{L} \text{ of finite rank}\bigr\}, \\ &= \sup\bigl\{i \in \mathbb{N}_0 \bigm| \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) \neq 0\bigr\}. \end{aligned}$$

Proposition 2.2.7. *Let X be a noetherian scheme such that $\mathfrak{Coh}(X)$ has enough locally frees. Let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathcal{F}$ is of finite Gorenstein homological dimension, say g, if and only if* $\operatorname{Ker} \partial_g$ *in any resolution of $\mathcal{F}$ by totally reflexive modules*

$$\mathcal{G}^\bullet\colon \quad \cdots \to \mathcal{G}_i \xrightarrow{\partial_i} \mathcal{G}_{i-1} \to \cdots \to \mathcal{G}_1 \xrightarrow{\partial_1} \mathcal{G}_0 \to \mathcal{F} \to 0$$

is totally reflexive. In particular, the nth syzygy, $n \geqslant g$, of any locally free resolution of $\mathcal{F}$ is totally reflexive.

Proof. The 'if' part is easy. For the 'only if' part, assume that $n \geqslant g$ and set $\mathcal{G} := \operatorname{Ker} \partial_n$. By [AM], for any $x \in X$, $\mathcal{G}_x \in \mathrm{G}(\mathcal{O}_x)$. So by Corollary 2.1.8, $\mathcal{G} \in \mathrm{G}(X)$. $\square$

3. Tate cohomology sheaves over noetherian schemes

3.1. Tate cohomology

Let X be a noetherian scheme and $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite Gorenstein homological dimension, say g. Consider a locally free resolution

$$\cdots \to \mathcal{L}_{g+1} \to \mathcal{L}_g \to \mathcal{L}_{g-1} \to \mathcal{L}_{g-2} \to \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

of $\mathcal{F}$, where as usual $\mathcal{L}_i$'s are of finite rank, and $\mathcal{G} = \operatorname{Ker}(\mathcal{L}_{g-1} \to \mathcal{L}_{g-2})$ belongs to $\mathrm{G}(X)$. Hence there exists a $\mathcal{H}om(\ ,\mathcal{O}_X)$-exact exact sequence

$$\mathcal{T}^\bullet\colon \quad \cdots \to \mathcal{T}_{g+2} \to \mathcal{T}_{g+1} \to \mathcal{T}_g \to \mathcal{T}_{g-1} \to \cdots \to \mathcal{T}_1 \to \mathcal{T}_0 \to \mathcal{T}_{-1} \to \cdots$$

of locally free sheaves of finite rank such that $\mathcal{G} = \operatorname{Ker}(\mathcal{T}_{g-1} \to \mathcal{T}_{g-2})$. We always may choose $\mathcal{T}_i$ to be equal to $\mathcal{L}_i$, for $i \geqslant g$. In this case, we call $\mathcal{T}^\bullet$ a complete resolution of $\mathcal{F}$.

Now assume that $\mathcal{F}'$ is an $\mathcal{O}_X$-module admitting a locally free resolution. For each $n \in \mathbb{Z}$, we define a *Tate cohomology sheaf* of $\mathcal{F}$ and $\mathcal{F}'$ by the equality

$$\widehat{\mathcal{E}xt}^n(\mathcal{F}, \mathcal{F}') = \mathcal{H}_{-n}\mathcal{H}om(\mathcal{T}^\bullet, \mathcal{F}').$$

A natural question now is that whether this definition is independent of the choice of $\mathcal{T}^\bullet$. In the following we establish this fact.

It follows from the construction that for $i > g$, we can choose $\mathcal{T}_i$ to be equal to $\mathcal{L}_i$. This in view of [H, III.6.5], implies that for each $i > g$, we always have

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}, \ ) = \mathcal{E}xt^i(\mathcal{F}, \ ).$$

Hence for $i > g$, Tate cohomology sheaves are well defined. Now let i be an arbitrary integer. Since $\mathcal{F}'$ has a locally free resolution we may consider a short exact sequence

$$0 \to \mathcal{K}_0 \to \mathcal{L}'_0 \to \mathcal{F}' \to 0,$$

in which $\mathcal{L}'_0$ is a locally free $\mathcal{O}_X$-module. Since $\mathcal{T}_i$ is locally free, the functor $\mathcal{H}om(\mathcal{T}_i, \ )$ is exact. So we get a short exact sequence of complexes

$$0 \to \mathcal{H}om(\mathcal{T}^\bullet, \mathcal{K}_0) \to \mathcal{H}om(\mathcal{T}^\bullet, \mathcal{L}'_0) \to \mathcal{H}om(\mathcal{T}^\bullet, \mathcal{F}') \to 0.$$

The associated long exact sequence of cohomology groups, in view of the fact that the middle complex is acyclic, gives the isomorphism

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{F}') \cong \widehat{\mathcal{E}xt}^{i+1}(\mathcal{F}, \mathcal{K}_0).$$

By continuing in this way, considering the short exact sequences

$$0 \to \mathcal{K}_j \to \mathcal{L}'_j \to \mathcal{K}_{j-1} \to 0$$

and following the same argument, we get the isomorphisms

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{F}') \cong \widehat{\mathcal{E}xt}^{i+j}(\mathcal{F}, \mathcal{K}_{j-1}).$$

But for j large enough, such that $i + j > g$, we have

$$\widehat{\mathcal{E}xt}^{i+j}(\mathcal{F}, \mathcal{K}_{j-1}) \cong \mathcal{E}xt^{i+j}(\mathcal{F}, \mathcal{K}_{j-1}).$$

And so

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{F}') \cong \mathcal{E}xt^{i+j}(\mathcal{F}, \mathcal{K}_{j-1}).$$

This means that if we have been started with any totally acyclic complex $\mathcal{T}'$ such that $\mathcal{G}$ is a kernel of it, we would get the same cohomology sheaves. In fact, we have shown that these cohomology sheaves are independent of the choice of complete resolutions of $\mathcal{F}$.

Moreover, one can see that for any sheaf $\mathcal{F}$ of finite Gorenstein homological dimension, $\widehat{\mathcal{E}xt}^n(\mathcal{F},\ ):\mathcal{L}(\mathcal{O}_X)\to\mathfrak{Mod}(X)$ is a covariant functor, where $\mathcal{L}(\mathcal{O}_X)$ denotes the full subcategory of $\mathfrak{Mod}(X)$ whose objects admit locally free resolution.

Following corollary is an easy consequence of the above discussion.

Corollary 3.1.1. *Assume that X is a noetherian scheme, that* $\operatorname{Ghd}\mathcal{F}<\infty$ *and that $\mathcal{F}'$ is a coherent sheaf admitting a locally free resolution. Then $\widehat{\mathcal{E}xt}^n(\mathcal{F},\mathcal{F}')$ is coherent, for all $n\in\mathbb{Z}$.*

Following lemma can be proved by induction on the homological dimension of $\mathcal{F}$. So we omit the proof.

Lemma 3.1.2. *Let*

$$\mathcal{T}^\bullet:\quad \cdots\to\mathcal{T}_2\to\mathcal{T}_1\to\mathcal{T}_0\to\mathcal{T}_{-1}\to\mathcal{T}_{-2}\to\cdots$$

be a totally acyclic complex. Then for any coherent sheaf $\mathcal{F}$ of finite homological dimension the complex $\mathcal{H}om(\mathcal{T}^\bullet,\mathcal{F})$ is exact.

Proposition 3.1.3. *Let X be a noetherian scheme and $\mathcal{F}$ be a coherent sheaf with* $\operatorname{hd}\mathcal{F}<\infty$. *Then*

$$\widehat{\mathcal{E}xt}^n(\mathcal{F},\ )=0=\widehat{\mathcal{E}xt}^n(\ ,\mathcal{F}).$$

Proof. Since $\operatorname{hd}\mathcal{F}=h<\infty$, we may choose a locally free resolution of $\mathcal{F}$ of length h. So $\mathcal{T}$ can be chosen to be the zero complex. Hence $\widehat{\mathcal{E}xt}^n(\mathcal{F},\ )=0$. The vanishing of $\widehat{\mathcal{E}xt}^n(\ ,\mathcal{F})$ follows from the above lemma. □

Using standard techniques in homological algebra, one can prove the following result.

Proposition 3.1.4. *Let X be a noetherian scheme. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite Gorenstein homological dimension and $0\to\mathcal{K}\to\mathcal{K}'\to\mathcal{K}''\to 0$ be a short exact sequence of $\mathcal{O}_X$-modules having locally free resolutions. Then there is a long exact sequence*

$$\cdots\longrightarrow\widehat{\mathcal{E}xt}^n(\mathcal{F},\mathcal{K})\longrightarrow\widehat{\mathcal{E}xt}^n(\mathcal{F},\mathcal{K}')\longrightarrow\widehat{\mathcal{E}xt}^n(\mathcal{F},\mathcal{K}'')$$

$$\longrightarrow\widehat{\mathcal{E}xt}^{n+1}(\mathcal{F},\mathcal{K})\longrightarrow\widehat{\mathcal{E}xt}^{n+1}(\mathcal{F},\mathcal{K}')\longrightarrow\cdots$$

of Tate cohomology sheaves.

We need the notion of Tate cohomology $\widehat{\operatorname{Ext}}_R$ for modules in our next result. So let us quote its definition from [AM]. Let M be a finite R-module of finite G-dimension. Choose a complete resolution $\mathbf{T}$ of M [AM, 3.1]. By this we mean a $\operatorname{Hom}(\ ,R)$-exact exact sequence of projectives such that for $i\gg 0$, it coincides with a projective resolution of M. Then, for each R-module N and for each $n\in\mathbb{Z}$, we define a Tate cohomology group by the equality

$$\widehat{\operatorname{Ext}}_R^n(M,N)=\operatorname{H}_{-n}\operatorname{Hom}_R(\mathbf{T},N).$$

Proposition 3.1.5. *Let X be a scheme of finite dimension. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite Gorenstein homological dimension. Then for any sheaf $\mathcal{F}'$ that admits a locally free resolution and any $x \in X$, we have*

$$\big(\widehat{\mathcal{E}xt}^n(\mathcal{F}, \mathcal{F}')\big)_x \cong \widehat{\operatorname{Ext}}^n_{\mathcal{O}_x}\big(\mathcal{F}_x, \mathcal{F}'_x\big).$$

Proof. Let $\mathcal{T}^\bullet$ be a complete resolution of $\mathcal{F}$. So $\mathcal{T}^\bullet_x$, for any $x \in X$, is a complete resolution of the $\mathcal{O}_x$-module $\mathcal{F}_x$. Since $\mathcal{T}_i$, for any integer i, is of finite rank, $\mathcal{H}om(\mathcal{T}^\bullet, \mathcal{F}')_x \cong \operatorname{Hom}(\mathcal{T}^\bullet_x, \mathcal{F}'_x)$. The result now follows. $\square$

Theorem 3.1.6. *Let X be a noetherian scheme of finite dimension. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite Gorenstein homological dimension. Then the following are equivalent.*

(i) $\operatorname{hd} \mathcal{F} < \infty$.
(ii) $\widehat{\mathcal{E}xt}^n(\mathcal{F}, \ ) = 0$, *for some* $n \in \mathbb{Z}$.
(iii) $\widehat{\mathcal{E}xt}^n(\mathcal{F}, \ ) = 0$, *for all* $n \in \mathbb{Z}$.
(iv) $\widehat{\mathcal{E}xt}^n(\ , \mathcal{F}) = 0$, *for some* $n \in \mathbb{Z}$.
(v) $\widehat{\mathcal{E}xt}^n(\ , \mathcal{F}) = 0$, *for all* $n \in \mathbb{Z}$.
(vi) $\widehat{\mathcal{E}xt}^0(\mathcal{F}, \mathcal{F}) = 0$.

Proof. In view of the previous results we just need to prove the implication (vi) $\Rightarrow$ (i). So assume that $\widehat{\mathcal{E}xt}^0(\mathcal{F}, \mathcal{F}) = 0$. It follows from the above proposition that for any $x \in X$, $\widehat{\operatorname{Ext}}^0(\mathcal{F}_x, \mathcal{F}_x) = 0$. So, by [AM, 5.9], $\operatorname{pd}_{\mathcal{O}_x} \mathcal{F}_x < \infty$, for any $x \in X$. The result now follows from the equality $\operatorname{hd} \mathcal{F} = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathcal{F}_x$. $\square$

Remark 3.1.7. The notion of Tate resolutions over an exterior algebra E is studied in [EFS, Section 4], as a part of the BGG correspondence, where coherent sheaves on P^n are represented by infinite exact sequences over an exterior algebra E. Since E is both projective and injective over itself, these Tate resolutions are totally acyclic and are uniquely determined by any of their kernels. But the way that this paper relates to the Tate resolutions is completely different with their point of view.

3.2. Alternative approach

We begin this subsection by recalling the notion of orthogonality. Let $(X, \mathcal{O}_X)$ be a scheme. For any class Θ of $\mathcal{O}_X$-modules, we let $\Theta^\perp$ denote the set of $\mathcal{O}_X$-modules orthogonal to Θ with respect to $\mathcal{E}xt^1$. Hence

$$\Theta^\perp = \big\{\mathcal{F}' \in \mathfrak{Mod}(X) \,\big|\, \mathcal{E}xt^1(\mathcal{F}, \mathcal{F}') = 0 \text{ for all } \mathcal{F} \in \Theta\big\}.$$

Definition 3.2.1. Let X be a noetherian scheme and $\mathcal{K}$ be an $\mathcal{O}_X$-module. By a ($\mathfrak{Coh}(X)^\perp$-) complete coresolution of $\mathcal{K}$ we mean an exact complex

$$\mathcal{T}^\bullet\colon \quad \cdots \to \mathcal{T}_2 \to \mathcal{T}_1 \to \mathcal{T}_0 \to \mathcal{T}_{-1} \to \mathcal{T}_{-2} \to \cdots$$

of modules in $\mathfrak{Coh}(X)^\perp$ such that for $i \ll 0$ it is coincide with an injective resolution of $\mathcal{K}$.

It is clear that injective sheaves belong to $\mathfrak{Coh}(X)^{\perp}$. Moreover, injective objects of $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on X, are in $\mathfrak{Coh}(X)^{\perp}$. In particular sheaves whose their stalks are injective are in $\mathfrak{Coh}(X)^{\perp}$.

Example 3.2.2. Let $\mathcal{L}^{\bullet}$ be a complete resolution of $\mathcal{F}$ and $\mathcal{I}$ be an arbitrary injective sheaf. Then $\mathcal{H}om(\mathcal{L}^{\bullet}, \mathcal{I})$ is a complete coresolution of $\mathcal{H}om(\mathcal{F}, \mathcal{I})$.

The idea of the following construction is taken from [GG, 4.1].

Proposition 3.2.3. *Let X be a locally Gorenstein scheme of dimension d. Let $\mathcal{K}$ be an $\mathcal{O}_X$-module admitting a locally free resolution. Then $\mathcal{K}$ has a complete coresolution.*

Proof. Let

$$\cdots \to \mathcal{L}_2 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{K} \to 0$$

be a locally free resolution of $\mathcal{K}$. Let $\mathcal{E}^{\bullet}$ and $\mathcal{E}_i^{\bullet}$ denote the injective resolution of $\mathcal{K}$ and $\mathcal{L}_i$, respectively. So we get a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{L}_2 \arrow[r] \arrow[d] & \mathcal{L}_1 \arrow[r] \arrow[d] & \mathcal{L}_0 \arrow[r] \arrow[d] & \mathcal{K} \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathcal{E}_2^{\bullet} \arrow[r] & \mathcal{E}_1^{\bullet} \arrow[r] & \mathcal{E}_0^{\bullet} \arrow[r] & \mathcal{E}^{\bullet} \arrow[r] & 0.
\end{tikzcd}$$

From this diagram we get an exact sequence

$$\Omega^{\bullet}: \quad \cdots \to \Omega^d \mathcal{L}_2 \to \Omega^d \mathcal{L}_1 \to \Omega^d \mathcal{L}_0 \to \mathcal{E}^d \to \mathcal{E}^{d+1} \to \mathcal{E}^{d+2} \to \cdots,$$

in which $\Omega^d \mathcal{L}_i$, for any integer i, denotes the sheaf $\operatorname{Im}(\mathcal{E}_i^{d-1} \to \mathcal{E}_i^d)$. We claim that this resolution is a complete coresolution of $\mathcal{K}$. To prove the claim, we have to show that for any coherent sheaf $\mathcal{F}$ and any integer $j \geqslant 0$, $\mathcal{E}xt^i(\mathcal{F}, \Omega^d \mathcal{L}_j) = 0$, for all $i > 0$. Consider the resolution

$$0 \to \mathcal{L}_j \to \mathcal{E}_j^0 \to \mathcal{E}_j^1 \to \cdots \to \mathcal{E}_j^{d-1} \to \Omega^d \mathcal{L}_j \to 0,$$

in which $\mathcal{E}_j^i$'s are injective. Let $\mathcal{F}$ be a coherent sheaf. By breaking up this sequence and applying the functor $\mathcal{E}xt(\mathcal{F}, \ )$ on these short exact sequences, we get a series of isomorphisms

$$\mathcal{E}xt^i\big(\mathcal{F}, \Omega^d \mathcal{L}_j\big) \cong \mathcal{E}xt^{i+1}\big(\mathcal{F}, \Omega^{d-1} \mathcal{L}_j\big) \cong \cdots \cong \mathcal{E}xt^{d+i}(\mathcal{F}, \mathcal{L}_j).$$

Let $x \in X$. Since $\mathcal{F}$ is coherent, by [H, III.6.8], $(\mathcal{E}xt^{d+i}(\mathcal{F}, \mathcal{L}_j))_x \cong \operatorname{Ext}_{\mathcal{O}_x}^{d+i}(\mathcal{F}_x, (\mathcal{L}_j)_x)$. But since X is a locally Gorenstein scheme of dimension d, $\mathcal{O}_x$ is a Gorenstein ring of Krull dimension less than or equal to d. Moreover $(\mathcal{L}_j)_x$ is a free $\mathcal{O}_x$-module and so it is of finite injective dimension. Hence $\operatorname{Ext}_{\mathcal{O}_x}^{d+i}(\mathcal{F}_x, (\mathcal{L}_j)_x) = 0$. This implies the result. $\square$

Example 3.2.4. By the above proposition, any sheaf over an affine scheme $X = \operatorname{Spec}(A)$, where A is a Gorenstein ring of finite dimension, admit a complete coresolution.

Example 3.2.5. Let X be an arithmetically Gorenstein subscheme of $\mathbb{P}_k^n$. By [H, III.5.18] any coherent sheaf $\mathcal{F}$ on X admits a resolution by dissocié sheaves. So, one can apply the same argument as in the proof of the above proposition, to deduce that $\mathcal{F}$ admits a complete coresolution. We just need to explain the fact that the dth cosyzygy of any injective coresolution of $\mathcal{O}_X(n)$ belongs to $\mathfrak{Coh}(X)^{\perp}$, where $d = \dim S(X)$. To see this, let $\mathcal{F}$ be an arbitrary coherent sheaf over X. Since $\mathcal{O}_X(n)$ is locally free and for any integer $i \geqslant 0$,

$$\mathcal{E}xt^i\big(\mathcal{F}, \mathcal{O}_X(n)\big) \cong \mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{O}_X(n),$$

it is enough to prove the claim for the dth cosyzygy of an injective coresolution of $\mathcal{O}_X$. So let

$$0 \to \mathcal{O}_X \to \mathcal{I}^0 \xrightarrow{\partial^0} \mathcal{I}^1 \to \cdots \to \mathcal{I}^{d-1} \xrightarrow{\partial^{d-1}} \mathcal{I}^d \to \cdots$$

be an injective coresolution of $\mathcal{O}_X$. For $i > 0$, set $\mathcal{L}^i := \operatorname{Coker} \partial^{i-1}$. So we get short exact sequences

$$0 \to \mathcal{L}^{i-1} \to \mathcal{I}^{i-1} \to \mathcal{L}^i \to 0.$$

Here $\mathcal{L}^0 = \mathcal{O}_X$. By applying the functor $\mathcal{E}xt(\mathcal{F},\ )$ on these sequences we get $\mathcal{E}xt^1(\mathcal{F}, \mathcal{L}^d) \cong \mathcal{E}xt^{d+1}(\mathcal{F}, \mathcal{O}_X)$. But by Theorem 2.2.5, $\operatorname{Ghd}\mathcal{F} \leqslant d$ and hence $\mathcal{E}xt^{d+1}(\mathcal{F}, \mathcal{O}_X) = 0$. Therefore $\mathcal{L}^d \in \mathfrak{Coh}(X)^{\perp}$.

Remark 3.2.6. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module in $\mathrm{G}(X)$. So there exists a short exact sequence $0 \to \mathcal{F} \to \mathcal{L} \to \mathcal{F}' \to 0$, where $\mathcal{L}$ is locally free and $\mathcal{F}' \in \mathrm{G}(X)$. But complete resolution of $\mathcal{F}$ and complete resolution of $\mathcal{F}'$ are the same up to shifting by 1. So for any $\mathcal{O}_X$-module $\mathcal{K}$ and any integer i, we have

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{K}) \cong \widehat{\mathcal{E}xt}^{i+1}(\mathcal{F}', \mathcal{K}).$$

We use this fact in the proof of our next theorem.

Lemma 3.2.7. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module of finite Gorenstein homological dimension. Then there exists a short exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{L} \to \mathcal{F} \to 0$$

of $\mathcal{O}_X$-modules such that $\mathcal{L}$ is locally free and

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}', \mathcal{K}) \cong \widehat{\mathcal{E}xt}^{i+1}(\mathcal{F}, \mathcal{K}).$$

Proof. Since $\operatorname{Ghd}\mathcal{F} = g < \infty$, there exists a locally free resolution

$$\cdots \to \mathcal{L}_g \to \mathcal{L}_{g-1} \to \mathcal{L}_{g-2} \to \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0$$

of $\mathcal{F}$ such that $\mathcal{G} = \operatorname{Ker}(\mathcal{L}_{g-1} \to \mathcal{L}_{g-2}) \in \mathrm{G}(X)$. Let $\mathcal{T}^{\bullet}$ be a totally acyclic complex related to $\mathcal{F}$. Set $\mathcal{F}' = \operatorname{Ker}(\mathcal{L}_0 \to \mathcal{F})$. So we get a short exact sequence $0 \to \mathcal{F}' \to \mathcal{L}_0 \to \mathcal{F} \to 0$, such that $\operatorname{Ghd}\mathcal{F}' = g - 1$. But in this situation, $\Sigma^{-1}\mathcal{T}^{\bullet}$, which is the shifting of $\mathcal{T}^{\bullet}$ one step to the left, is the totally acyclic complex related to the $(g-1)$th syzygy of $\mathcal{F}'$. So it is a complete resolution of $\mathcal{F}'$. Hence

$$\widehat{\mathcal{E}xt}^i(\mathcal{F}',\mathcal{K}) \cong \mathcal{H}_{-i}\mathcal{H}om\big(\Sigma^{-1}\mathcal{T}^\bullet,\mathcal{K}\big) \cong \mathcal{H}_{-i-1}\mathcal{H}om\big(\mathcal{T}^\bullet,\mathcal{K}\big) \cong \widehat{\mathcal{E}xt}^{i+1}(\mathcal{F},\mathcal{K}). \qquad \square$$

Theorem 3.2.8. *Let X be a noetherian scheme and let $\mathcal{K}$ be an $\mathcal{O}_X$-module admitting a complete coresolution $\mathcal{T}^\bullet$. Then for any $\mathcal{O}_X$-module $\mathcal{F}$ of finite Gorenstein homological dimension, $\widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K}) = \mathcal{H}_{-i}\mathcal{H}om(\mathcal{F},\mathcal{T}^\bullet)$.*

Proof. Let $\mathcal{T}^\bullet$: $\cdots \to \mathcal{T}_2 \to \mathcal{T}_1 \to \mathcal{T}_0 \to \mathcal{T}_{-1} \to \mathcal{T}_{-2} \to \cdots$ be an arbitrary complete coresolution of $\mathcal{K}$ and n be an integer such that for $i < n$, $\mathcal{T}^\bullet$ is compatible with an injective resolution of $\mathcal{K}$. Hence for $i > n$, $\widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K}) = \mathcal{E}xt^i(\mathcal{F},\mathcal{K}) \cong \mathcal{H}_{-i}\mathcal{H}om(\mathcal{F},\mathcal{T}^\bullet)$. Assume that $i \leqslant n$, and consider $\mathcal{H}_{-i}\mathcal{H}om(\mathcal{F},\mathcal{T}^\bullet)$. Suppose inductively, that $\operatorname{Ghd}\mathcal{F} = 0$. So there exists a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{L} \to \mathcal{F}' \to 0$$

of sheaves, where $\mathcal{L}$ is locally free and $\mathcal{F}' \in \mathrm{G}(X)$. Since $\mathcal{H}_i\mathcal{H}om(\mathcal{L},\mathcal{T}^\bullet) = 0$, for all integers i, $\mathcal{H}_i\mathcal{H}om(\mathcal{F},\mathcal{T}^\bullet) \cong \mathcal{H}_{i+1}\mathcal{H}om(\mathcal{F}',\mathcal{T}^\bullet)$. By continuing in this way, this time for $\mathcal{F}'$, we can increase the indices of the cohomology up to $n+1$, where it is equivalence with the usual $\mathcal{E}xt$. Now result follows in view of the above remark. Suppose inductively that $\operatorname{Ghd}\mathcal{F} > 0$. So there exists an exact sequence

$$0 \to \mathcal{H} \to \mathcal{L} \to \mathcal{F} \to 0$$

of sheaves in which $\mathcal{L}$ is locally free and $\operatorname{Ghd}\mathcal{H} = n-1$. Since for any integer i, $\mathcal{T}_i \in \mathfrak{Coh}(X)^\perp$, we get a short exact sequence of complexes

$$0 \to \mathcal{H}om\big(\mathcal{F},\mathcal{T}^\bullet\big) \to \mathcal{H}om\big(\mathcal{L},\mathcal{T}^\bullet\big) \to \mathcal{H}om\big(\mathcal{H},\mathcal{T}^\bullet\big) \to 0.$$

Taking the associated long exact sequence of cohomology groups, in view of the fact that $\mathcal{H}_{-i}\mathcal{H}om(\mathcal{L},\mathcal{T}^\bullet) = 0$, for all integers i, implies that $\mathcal{H}_{-i}\mathcal{H}om(\mathcal{F},\mathcal{T}^\bullet) \cong \mathcal{H}_{-i-1}\mathcal{H}om(\mathcal{H},\mathcal{T}^\bullet)$. By our induction assumption, $\mathcal{H}_{-i-1}\mathcal{H}om(\mathcal{H},\mathcal{T}^\bullet) \cong \widehat{\mathcal{E}xt}^{i+1}(\mathcal{H},\mathcal{K})$. By the above lemma, $\widehat{\mathcal{E}xt}^{i+1}(\mathcal{H},\mathcal{K}) \cong \widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K})$. Therefore $\mathcal{H}_{-i}\mathcal{H}om(\mathcal{F},\mathcal{T}^\bullet) \cong \widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K})$. This complete the inductive step and hence the proof. $\square$

By Theorem 2.2.3, over locally Gorenstein schemes of finite dimensions, in which $\mathfrak{Coh}(X)$ has enough locally frees, Gorenstein homological dimension of any coherent sheaf is finite. Moreover, by Proposition 3.2.3, over such schemes every $\mathcal{O}_X$-module $\mathcal{K}$ has a complete coresolution. So we have the following corollary.

Corollary 3.2.9. *Let X be a locally Gorenstein scheme of finite dimension such that $\mathfrak{Coh}(X)$ has enough locally frees. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any $\mathcal{O}_X$-module $\mathcal{K}$,*

$$\widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K}) = \mathcal{H}_{-i}\mathcal{H}om\big(\mathcal{F},\mathcal{T}^\bullet\big),$$

where $\mathcal{T}^\bullet$ is a complete coresolution of $\mathcal{K}$.

Corollary 3.2.10. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module such that both injective dimension and Gorenstein homological dimension of $\mathcal{F}$ is finite. Then homological dimension of $\mathcal{F}$ is finite.*

Proof. Since injective dimension of $\mathcal{F}$ is finite, the zero complex can be considered as a complete coresolution of it. By the above theorem, since Gorenstein homological dimension of $\mathcal{F}$ is finite,

$$\mathcal{H}_{-n}\big(\mathcal{H}om(\mathcal{F},0)\big)\cong\widehat{\mathcal{E}xt}^{n}(\mathcal{F},\mathcal{F}),$$

for any integer n. In particular, $\widehat{\mathcal{E}xt}^{0}(\mathcal{F},\mathcal{F})=0$. The result now follows from Theorem 3.1.6. $\square$

4. Complete cohomology

In this section, we study complete cohomology in the category of sheaves of modules over a ringed space $(X,\mathcal{O}_X)$ and compare it with the above theory of Tate cohomology. As application, we give some criteria for the Gorensteinness of schemes in terms of the vanishing or coherentness of the complete cohomology sheaves. First, we review the construction of the complete cohomology theory for modules over an associative ring. It will help us in better understanding the theory for shoves of modules over a ringed space. Moreover, we use these results in proving our main results.

4.1. Complete cohomology of modules

Let R be an associative ring and let $M\to I^\bullet$ and $N\to J^\bullet$ be injective coresolutions of the (left) R-modules M and N, respectively. For any integer n, let $\mathrm{Hom}_R(I^\bullet,J^\bullet)_n$ denote the abelian group of all homomorphisms $\varphi: I^\bullet\to J^\bullet$ of degree n. It is isomorphic to $\prod_{i\in\mathbb{Z}}\mathrm{Hom}_R(I_i,J_{i+n})$. Denote by $\mathrm{Hom}_R(I^\bullet,J^\bullet)$ the complex of R-modules with $\mathrm{Hom}_R(I^\bullet,J^\bullet)_n$ as nth components and differentials $\partial(\varphi)=\partial^{\mathbf{J}}\varphi+(-1)^{|\varphi|}\varphi\partial^{\mathbf{I}}$, where $|\varphi|$ denotes the degree of φ. Here we use subscripts to grade the complexes.

A morphism $f\in\mathrm{Hom}_R(I^\bullet,J^\bullet)$ is called bounded if $f_i=0$, for all $i\ll 0$. Let $\mathrm{Hom}_R(I^\bullet,J^\bullet)_b$ denote the subcomplex of the complex $\mathrm{Hom}_R(I^\bullet,J^\bullet)$ of bounded morphisms. So

$$\big(\mathrm{Hom}_R\big(I^\bullet,J^\bullet\big)_b\big)_n=\bigoplus_{i\in\mathbb{Z}}\mathrm{Hom}_R(I_i,J_{i+n}).$$

We let $\widetilde{\mathrm{Hom}}_R(M,N)$ denote the quotient complex $\mathrm{Hom}_R(I^\bullet,J^\bullet)/\mathrm{Hom}_R(I^\bullet,J^\bullet)_b$. Now passing on to cohomology we get the complete cohomology of M and N over R, denoted $\widetilde{\mathrm{ext}}_R(M,N)$. So, for any integer n, $\widetilde{\mathrm{ext}}^n_R(M,N)=\mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(I^\bullet,J^\bullet))$.

It is easy to check that $\widetilde{\mathrm{ext}}^n_R(M,N)$ is independent of the choice of injective coresolutions of M and N. In fact, the assignment $(M,N)\to\widetilde{\mathrm{ext}}_R(M,N)$ defines a cohomological functor from $\mathcal{C}(R)$, the category of R-modules, to $\mathcal{C}(\mathbb{Z})$, contravariant in the first and covariant in the second variable. For more details, see [N].

Remark 4.1.1. The exact sequence

$$0\to\mathrm{Hom}_R\big(I^\bullet,J^\bullet\big)_b\to\mathrm{Hom}_R\big(I^\bullet,J^\bullet\big)\to\widetilde{\mathrm{Hom}}_R\big(I^\bullet,J^\bullet\big)\to 0,$$

of complexes induces, for any integer n, an exact sequence of cohomological modules

$$\cdots \to \overline{\mathrm{ext}}_R^n(M,N) \to \mathrm{Ext}_R^n(M,N) \overset{\beta_R^i(M,N)}{\longrightarrow} \widetilde{\mathrm{ext}}_R^n(M,N) \to \overline{\mathrm{ext}}_R^{n+1}(M,N) \to \cdots,$$

where $\overline{\mathrm{ext}}_R^n(M,N)$ is the $(-n)$th cohomology group of the complex $\mathrm{Hom}_R(I^\bullet, J^\bullet)_b$.

Remark 4.1.2. Let N be an R-module. By an injective complete coresolution of N we mean an exact complex $T^\bullet$ of injective R-modules, indexed by integers, such that $T^\bullet$ coincides with an injective coresolution of N for $n \ll 0$ and $\mathrm{Hom}_R(J, T^\bullet)$ is exact, for all injective R-modules J.

Now suppose that $T^\bullet$ is an injective complete coresolution of N. In [AS], we define a cohomology theory by the equality $\widehat{\mathrm{Ext}}_R^n(M,N) = \mathrm{H}^n \mathrm{Hom}_R(M, T^\bullet)$. It is shown that $\widehat{\mathrm{Ext}}_R^n : \mathcal{C}(R) \times \widetilde{\mathcal{GI}}(R) \to \mathcal{C}(\mathbb{Z})$ is a bifunctor, where $\widetilde{\mathcal{GI}}(R)$ denotes the full subcategory of $\mathcal{C}(R)$ whose objects are modules of finite Gorenstein injective dimension. Moreover, there is a morphism of functors $\partial_R^n : \mathrm{Ext}_R^n \to \widehat{\mathrm{Ext}}_R^n$ such that $\widehat{\mathrm{Ext}}_R^n$ and ∂_R^n are independent of the choice of resolutions and liftings.

When N is a module of finite Gorenstein injective dimension, then there exists a natural isomorphism of cohomology functors $\widehat{\mathrm{Ext}}_R^n(M,N) \cong \widetilde{\mathrm{ext}}_R^n(M,N)$, compatible with the natural maps from $\mathrm{Ext}_R^n(M,N)$.

4.2. Complete cohomology of sheaves

Let $\mathcal{F} \to \mathcal{I}^\bullet$ and $\mathcal{G} \to \mathcal{J}^\bullet$ be injective coresolutions of the $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, respectively. Denote by $\mathrm{Hom}(\mathcal{I}^\bullet, \mathcal{J}^\bullet)$, the complex Hom.

Let $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)$ denote the sheaf of complexes of groups defined by

$$U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}\big(\mathcal{I}^\bullet|_U, \mathcal{J}^\bullet|_U\big).$$

Since $\mathcal{J}^\bullet$ is a complex of injective $\mathcal{O}_X$-modules, the induced map $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet) \to \mathcal{H}om(\mathcal{F}, \mathcal{J}^\bullet)$ is quasiisomorphism and so for all integers n,

$$\mathcal{H}_{-n}\big(\mathcal{H}om\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big)\big) \cong \mathcal{H}_{-n}\big(\mathcal{H}om\big(\mathcal{F}, \mathcal{J}^\bullet\big)\big) = \mathcal{E}xt^n(\mathcal{F}, \mathcal{G}).$$

Note that $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_n = \prod_{i\in\mathbb{Z}} \mathcal{H}om(\mathcal{I}_i, \mathcal{J}_{i+n})$. We denote by $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_b$ the subsheaf of $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)$ whose nth term is $(\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_b)_n = \bigoplus_{i\in\mathbb{Z}} \mathcal{H}om(\mathcal{I}_i, \mathcal{J}_{i+n})$. We call $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_b$ the subsheaf of bounded morphisms. Finally, we define the subsheaf $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_{lb}$ of locally bounded morphisms to be a subsheaf consists of locally bounded morphisms on X, that is, there exists an open cover U_i of X such that

$$\mathcal{H}om\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big)_{lb}|_{U_i} = \mathcal{H}om\big(\mathcal{I}^\bullet|_{U_i}, \mathcal{J}^\bullet|_{U_i}\big)_b.$$

It is clear that if X is noetherian then $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_{lb} = \mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_b$.

Considering the subsheaf $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_{lb}$ of $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)$, set

$$\widetilde{\mathcal{H}om}\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big) = \mathcal{H}om\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big)/\mathcal{H}om\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big)_{lb},$$

which is again a complex of $\mathcal{O}_X$-modules. Following Vogel, we define the nth complete cohomology of $\mathcal{F}$ and $\mathcal{G}$ over X to be

$$\widetilde{\mathcal{E}xt}^n(\mathcal{F}, \mathcal{G}) = \mathcal{H}_{-n}\big(\widetilde{\mathcal{H}om}\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big)\big).$$

One can verify by a routine check analogous to ordinary cohomology that $\widetilde{\mathcal{E}xt}^n(\mathcal{F},\mathcal{G})$ is independent of the choice of injective coresolutions of $\mathcal{F}$ and $\mathcal{G}$. In fact, the assignment $(\mathcal{F},\mathcal{G}) \mapsto \widetilde{\mathcal{E}\mathrm{xt}}(\mathcal{F},\mathcal{G})$ defines a cohomological functor, from the category of $\mathcal{O}_X$-modules to itself, contravariant in $\mathcal{F}$ and covariant in $\mathcal{G}$. So we have long exact sequences of $\widetilde{\mathcal{E}\mathrm{xt}}$ sheaves associated to short exact sequences of $\mathcal{O}_X$-modules in both variables.

4.2.1. The exact sequence

$$0 \to \mathcal{H}om\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big)_{lb} \to \mathcal{H}om\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big) \to \widetilde{\mathcal{H}om}\big(\mathcal{I}^\bullet, \mathcal{J}^\bullet\big) \to 0$$

of complexes of $\mathcal{O}_X$-modules, induces, for any integer $n \geqslant 0$, an exact sequence of cohomological functors

$$\cdots \to \overline{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G}) \to \mathcal{E}xt^n(\mathcal{F},\mathcal{G}) \to \widetilde{\mathcal{E}xt}^n(\mathcal{F},\mathcal{G}) \to \overline{\mathcal{E}\mathrm{xt}}^{n+1}(\mathcal{F},\mathcal{G}) \to \cdots,$$

where $\overline{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G})$ is the nth cohomology of the complex $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)_{lb}$.

It follows from the definition that if $\mathcal{F}$ or $\mathcal{G}$ has injective coresolution of finite length, then $\widetilde{\mathcal{E}xt}^n(\mathcal{F},\mathcal{G}) = 0$, for all $n \in \mathbb{Z}$ and hence $\overline{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G}) \cong \mathcal{E}xt^n(\mathcal{F},\mathcal{G})$.

Remark 4.2.2. Let X be a noetherian scheme. Assume that $\mathcal{F}$ and $\mathcal{G}$ are both coherent. Then for all integers $n \geqslant 0$, $\mathcal{E}xt^n(\mathcal{F},\mathcal{G})$ is coherent. Hence, for each $n \in \mathbb{Z}$, $\widetilde{\mathcal{E}xt}^n(\mathcal{F},\mathcal{G})$ is coherent if and only if $\overline{\mathcal{E}\mathrm{xt}}^{n+1}(\mathcal{F},\mathcal{G})$ is coherent.

5. Gorensteinness

5.1. Gorensteinness of local rings

We aim to give a description of Gorenstein local rings in terms of the vanishing of complete cohomology. From now on, we assume that $(R,\mathfrak{m})$ is a commutative noetherian local ring with residue field k. The results in this subsection, which are of independent interest, will be used in the next subsection.

Lemma 5.1.1. *Let M be an Artinian R-module. If $\mathrm{Tor}_i^R(M,k) = 0$ for all $i > \dim R$, then* $\mathrm{fd}_R M < \infty$.

Proof. It is enough to show that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i > \dim R$ and for all finite R-module N. To this end, we use induction on $\dim N$. Suppose first that $\dim N = 0$. So $\ell(N)$, the length of N, is finite, say s. We use induction on s, to prove the result in this case. For $s = 1$, there is nothing to prove. Let $s > 1$ and consider the short exact sequence

$$0 \to k \to N \to L \to 0,$$

where $\ell(L) = s - 1$. Now long exact sequence of 'Tor' modules arising from this short exact sequence, in view of the induction assumption, implies that $\mathrm{Tor}_i^R(M,N) = 0$, for all $i > \dim R$.

So assume that $\dim N > 0$ and the result has been proved for all modules of dimension less than $\dim N$. Consider the long exact sequence of 'Tor' modules arising from the short exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(N) \to N \to N/\Gamma_{\mathfrak{m}}(N) \to 0.$$

Since $\dim \Gamma_{\mathfrak{m}}(N) = 0$, by induction assumption, $\operatorname{Tor}_i^R(M, \Gamma_{\mathfrak{m}}(N)) = 0$, for all $i > \dim R$. So for $i > \dim R$, $\operatorname{Tor}_i^R(M, N) = 0$, if $\operatorname{Tor}_i^R(M, N/\Gamma_{\mathfrak{m}}(N)) = 0$. Hence, we may (and do) assume that N is $\mathfrak{m}$-torsion-free. So $\mathfrak{m}$ contains a non-zero divisor r on N. Consider the short exact sequence

$$0 \to N \xrightarrow{r} N \to N/rN \to 0,$$

to deduce, for any integer i, the exact sequence

$$\operatorname{Tor}_{i+1}^R(M, N/rN) \to \operatorname{Tor}_i^R(M, N) \xrightarrow{r} \operatorname{Tor}_i^R(M, N).$$

So, by induction assumption, for $i > \dim R$, we get an injection $\operatorname{Tor}_i^R(M, N) \xrightarrow{r} \operatorname{Tor}_i^R(M, N)$. But since $\operatorname{Tor}_i^R(M, N)$ is Artinian, the dual of Nakayama lemma implies that $\operatorname{Tor}_i^R(M, N) = 0$. The inductive step and hence the proof is now complete. $\square$

Corollary 5.1.2. *Let* $\operatorname{Ext}_R^i(E(R/\mathfrak{m}), k) = 0$, *for all* $i \gg 0$. *Then* R *is Gorenstein.*

Proof. In view of the canonical isomorphisms

$$\begin{aligned}\operatorname{Ext}_R^i\big(E(R/\mathfrak{m}), k\big) &\cong \operatorname{Ext}_R^i\big(E(R/\mathfrak{m}), \operatorname{Hom}_R\big(k, E(R/\mathfrak{m})\big)\big)\\ &\cong \operatorname{Hom}_R\big(\operatorname{Tor}_i^R\big(E(R/\mathfrak{m}), k\big), E(R/\mathfrak{m})\big),\end{aligned}$$

we may deduce that $\operatorname{Tor}_i^R(E(R/\mathfrak{m}), k) = 0$, for all $i > \dim R$. The previous lemma, implies that $\operatorname{fd}_R E(R/\mathfrak{m}) < \infty$. Hence R is Gorenstein. $\square$

In the following, we shall use D to denote the *Matlis dual* functor $\operatorname{Hom}_R(\ , E(R/\mathfrak{m}))$.

Proposition 5.1.3. *Let* M *be a finite and Matlis reflexive* R*-module. Then for any integer* $n \in \mathbb{Z}$, *there is an isomorphism of* k*-vector spaces*

$$\overline{\operatorname{ext}}_R^n(M, k) \cong \bigoplus_{i \in \mathbb{Z}} \operatorname{Ext}_R^i\big(E(R/\mathfrak{m}), k\big) \otimes_k \operatorname{Tor}_{i-n}^R\big(k, D(M)\big).$$

Proof. Let $k \to J^\bullet$ be an injective coresolution of k and consider a free resolution $F^\bullet \to D(M)$ of $D(M)$. Since M is Matlis reflexive, $D(F^\bullet)$ is an injective coresolution of M. There is a canonical morphism

$$\omega : \operatorname{Hom}_R\big(E(R/\mathfrak{m}), J^\bullet\big) \otimes_R F^\bullet \to \operatorname{Hom}_R\big(D\big(F^\bullet\big), J^\bullet\big),$$

where its image lies in $\operatorname{Hom}_R(D(F^\bullet), J^\bullet)_b$. Moreover, for each $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\big(\mathrm{Hom}_R\big(E(R/\mathfrak{m}), J^\bullet\big)\otimes_R F^\bullet\big)_n &\cong \bigoplus_{i+j=n} \mathrm{Hom}_R\big(E(R/\mathfrak{m}), J_i\big)\otimes_R F_j &(1)\\
&= \bigoplus_{i+j=n} \mathrm{Hom}_R\big(\mathrm{Hom}_R\big(F_j, E(R/\mathfrak{m})\big), J_j\big) &(2)\\
&= \bigoplus_{l\in\mathbb{Z}} \mathrm{Hom}_R\big(D(F)_l, J_{n+l}\big) &(3)\\
&= \big(\mathrm{Hom}_R\big(D\big(F^\bullet\big), J^\bullet\big)_b\big)_n. &(4)
\end{aligned}$$

So ω induces an isomorphism of complexes

$$\omega : \mathrm{Hom}_R\big(E(R/\mathfrak{m}), J^\bullet\big)\otimes_R F^\bullet \to \mathrm{Hom}_R\big(I^\bullet, J^\bullet\big)_b,$$

where $I^\bullet = D(F^\bullet)$ is the injective coresolution of M. Now let $P^\bullet \to E(R/\mathfrak{m})$ be a projective resolution of $E(R/\mathfrak{m})$. This gives quasiisomorphisms

$$\begin{aligned}
\mathrm{Hom}_R\big(E(R/\mathfrak{m}), J^\bullet\big)\otimes_R F^\bullet &\cong \mathrm{Hom}_R\big(P^\bullet, J^\bullet\big)\otimes_R F^\bullet\\
&= \mathrm{Hom}_R\big(P^\bullet, k\big)\otimes_R F^\bullet.
\end{aligned}$$

The last complex is clearly quasiisomorphic to the complex $\mathrm{Hom}_R(P^\bullet, k)\otimes_k (k\otimes_R F^\bullet)$. So, putting together the above results, we get quasiisomorphism

$$\mathrm{Hom}_R\big(I^\bullet, J^\bullet\big)_b \cong \mathrm{Hom}_R\big(P^\bullet, k\big)\otimes_k \big(k\otimes_R F^\bullet\big).$$

Now Künneth formula implies the desired isomorphism of k-vector spaces. □

Corollary 5.1.4. *Let M be finite and Matlis reflexive. If $\widetilde{\mathrm{ext}}^n_R(M, k)$ is finite, for some $n \in \mathbb{Z}$, then either R is Gorenstein or $\mathrm{id}_R M$ is finite.*

Proof. It follows from Remark 4.2.2 and our assumption that $\overline{\mathrm{ext}}^{n+1}_R(M, k)$ is of finite length. Hence $\mathrm{rank}_k\, \overline{\mathrm{ext}}^{n+1}_R(M, k)$ is finite. So by the previous proposition, either $\mathrm{Ext}^i_R(E(R/\mathfrak{m}), k) = 0$ for $i \gg 0$ or $\mathrm{Tor}^R_j(k, D(M)) = 0$ for $j \gg 0$. In the first case, we get $\mathrm{fd}_R\, E(R/\mathfrak{m}) < \infty$, which implies that R is Gorenstein, while in the second case, we deduce that $\mathrm{fd}_R\, D(M) < \infty$, which implies the finiteness of the injective dimension of M. □

In view of the above result and Bass theorem [Mat, Section 18], we can record the following corollary.

Corollary 5.1.5. *A local ring R is Cohen–Macaulay if and only if there exists a finite Matlis reflexive R-module M such that $\widetilde{\mathrm{ext}}^n_R(M, k)$ is finite, for some $n \in \mathbb{Z}$.*

Following theorem should be compared with [AV, 4.1].

Theorem 5.1.6. *For a commutative noetherian local ring R, the following conditions are equivalent.*

(i) *R is Gorenstein.*
(ii) $\widetilde{\operatorname{ext}}^n_R(M,N)$ *is finite, for all finite modules M and N and all positive integers n.*
(iii) $\widetilde{\operatorname{ext}}^n_R(k,k)$ *is finite, for some integer n.*

Proof. (i) $\Rightarrow$ (ii). Assume that R is Gorenstein. So by Remark 4.2.2, complete cohomology is isomorphic to the Tate cohomology. The result follows from [AS, 4.8].

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). This follows from Corollary 5.1.4. $\square$

5.2. Gorensteinness of schemes

In this subsection, $(X, \mathcal{O}_X)$ is a scheme. We aim to establish the fact that complete cohomology reflects Gorensteinness of the underlying scheme. We begin by introducing the notion of locally injective sheaves.

Definition 5.2.1. An $\mathcal{O}_X$-module $\mathcal{L}$ is called locally injective if X can be covered by open sets U for which $\mathcal{L}|_U$ is an injective $\mathcal{O}_X|_U$-module.

Clearly, injective sheaves are locally injective. Also, if $\mathcal{F}$ is locally free and $\mathcal{I}$ is an injective sheaf, then $\mathcal{H}om(\mathcal{F},\mathcal{I})$ is locally injective $\mathcal{O}_X$-module. This follows from [H, III.6.1].

Lemma 5.2.2. *Let $\mathcal{L}$ be a locally injective module. Then, for any $\mathcal{F} \in \mathfrak{Mod}(X)$, $\mathcal{E}xt^n(\mathcal{F},\mathcal{L}) = 0$, for all $n > 0$.*

Proof. By [H, III.6.2], for any open subset U of X we have

$$\mathcal{E}xt^n_X(\mathcal{F},\mathcal{L})|_U \cong \mathcal{E}xt^n_U(\mathcal{F}|_U,\mathcal{L}|_U).$$

Now since $\mathcal{L}$ is locally injective, there exists a covering of X by open subsets U_i, such that for each i, $\mathcal{L}|_{U_i}$ is an injective $\mathcal{O}_{U_i}$-module. Hence for any $\mathcal{F} \in \mathfrak{Mod}(X)$, $\mathcal{E}xt^n_X(\mathcal{F},\mathcal{L})|_{U_i} = 0$, for all integers $n > 0$. This implies that $\mathcal{E}xt^n_X(\mathcal{F},\mathcal{L}) = 0$, for all $n > 0$. $\square$

The above lemma shows that locally injective sheaves can be used to compute $\mathcal{E}xt$ sheaves. We shall use this fact latter.

Definition 5.2.3. Let $\mathcal{G}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{G}$ is of finite locally injective dimension if there exists a covering of X by open subsets U such that $\mathcal{G}|_U$, as $\mathcal{O}_X|_U$-module, has an injective coresolution of finite length. We denote this by $\operatorname{lid}(\mathcal{G}) < \infty$

Theorem 5.2.4. *For an $\mathcal{O}_X$-module $\mathcal{G}$, the following are equivalent.*

(i) $\operatorname{lid}(\mathcal{G}) < \infty$.
(ii) $\widetilde{\mathcal{E}\mathrm{xt}}^n(\ ,\mathcal{G}) = 0$, *for all integers n.*
(iii) $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{G},\ ) = 0$, *for all integers n.*
(iv) $\widetilde{\mathcal{E}\mathrm{xt}}^0(\mathcal{G},\mathcal{G}) = 0$.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ and $\mathcal{G} \to \mathcal{J}^\bullet$ be injective coresolutions of $\mathcal{F}$ and $\mathcal{G}$, respectively, where $\mathcal{F}$ is an arbitrary sheaf. Since $\mathrm{lid}(\mathcal{G}) < \infty$, any morphism in $\mathcal{H}om(\mathcal{I}^\bullet, \mathcal{J}^\bullet)$ (respectively $\mathcal{H}om(\mathcal{J}^\bullet, \mathcal{I}^\bullet)$) is locally bounded. This implies that the quotient complex should be zero and hence $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{F}, \mathcal{G}) = 0$ (respectively $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{G}, \mathcal{F}) = 0$), for all integers n. This implies (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (i). Since $\widetilde{\mathcal{E}\mathrm{xt}}^0(\mathcal{G}, \mathcal{G}) = 0$, it follows from the exact sequence of 4.2.1 that the morphism

$$\overline{\mathcal{E}\mathrm{xt}}^0(\mathcal{G}, \mathcal{G}) \xrightarrow{\bar{\gamma}} \mathcal{E}xt^0(\mathcal{G}, \mathcal{G})$$

is surjective. Consider the identity morphism $\mathrm{id} \in \mathcal{H}om(\mathcal{G}, \mathcal{G})(X)$. By [H, Ex. II.1.3(a)], there is a covering $\{U_i\}$ of X and the elements $\bar{g}_i \in \mathcal{H}_0(\mathcal{H}om(\mathcal{J}^\bullet, \mathcal{J}^\bullet)_{lb})(U_i)$ such that $\bar{\gamma}(\bar{g}_i) = \mathrm{id}|_{U_i}$ for all i. But $\bar{g}_i = g_i + \mathrm{Im}\,\bar{\partial}_1$, where $g_i \in (\mathrm{Hom}(\mathcal{J}^\bullet|_{U_i}, \mathcal{J}^\bullet|_{U_i})_b)_0$. So for all i, there exists an integer $j_i \ll 0$ such that $(g_i)_j = 0$ for all $j < j_i$. On the other hand, it follows from assumption that $\mathrm{id}|_{U_i} - \gamma(g_i) \in \mathrm{Im}(\partial_1|_{U_i})$, where

$$\partial_1|_{U_i} : \mathcal{H}om\big(\mathcal{J}^\bullet, \mathcal{J}^\bullet\big)_1(U_i) \to \mathcal{H}om\big(\mathcal{J}^\bullet, \mathcal{J}^\bullet\big)_0(U_i).$$

So there exists $h_i \in \mathrm{Hom}(\mathcal{J}^\bullet|_{U_i}, \mathcal{J}^\bullet|_{U_i})_1$ such that $\partial_1(h_i) = \mathrm{id}|_{U_i} - \gamma(g_i)$. Since g_i is bounded, for $j \ll 0$, $(\mathrm{id}|_{U_i} - \gamma(g_i))_j$ is the identity on $J^\bullet|_{U_i}$. So for all $j \ll 0$,

$$\partial_{j+1}^{\mathcal{J}^\bullet|_{U_i}}(h_i)_j - (h_i)_{j-1}\partial_j^{\mathcal{J}^\bullet|_{U_i}} = (\mathrm{id}|_{U_i})^{\mathcal{J}^\bullet|_{U_i}}.$$

But $\mathcal{K}er\,\partial_j^{\mathcal{J}^\bullet|_{U_i}} = \mathcal{I}m\,\partial_{j+1}^{\mathcal{J}^\bullet|_{U_i}}$. So the map $J_{j+1}^{\mathcal{J}^\bullet|_{U_i}} \to \mathcal{I}m\,\partial_{j+1}^{\mathcal{J}^\bullet|_{U_i}} \to 0$ splits. Hence $\mathcal{I}m\,\partial_{j+1}^{J^\bullet}|_{U_i}$ is injective. This shows that $\mathcal{G}|_{U_i}$ has an injective coresolution of finite length. Therefore $\mathrm{lid}(\mathcal{G}) < \infty$. $\square$

Towards the end of this subsection, we consider sheaves in $\mathfrak{Qco}(X)$, the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules. Moreover, we assume that X is noetherian. We know from [H, III.3.6] that this category has enough injectives. So we compute all the above three kinds of cohomology sheaves in $\mathfrak{Qco}(X)$. In order to avoid any confusion, we denote these new functors by putting a subscript $\mathfrak{Q}(X)$ next to them, so we write $\mathcal{E}xt_{\mathfrak{Q}(X)}$, $\widetilde{\mathcal{E}\mathrm{xt}}^1_{\mathfrak{Q}(X)}$ and $\overline{\mathcal{E}\mathrm{xt}}_{\mathfrak{Q}(X)}$. Moreover, locally injective dimension of $\mathcal{G}$ in $\mathfrak{Qco}(X)$ will be denoted by $\mathrm{lid}_{\mathfrak{Q}(X)}(\mathcal{G})$.

Theorem 5.2.5. *Let X be a noetherian scheme of finite dimension d. Then the following conditions are equivalent.*

(i) *X is locally Gorenstein.*
(ii) $\mathrm{lid}_{\mathfrak{Q}(X)}(\mathcal{O}_X) < \infty$.
(iii) $\widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(X)}(\ , \mathcal{O}_X) = 0$, *for all integers n.*
(iv) $\widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(X)}(\mathcal{O}_X, \ ) = 0$, *for all integers n.*
(v) $\widetilde{\mathcal{E}\mathrm{xt}}^0_{\mathfrak{Q}(X)}(\mathcal{O}_X, \mathcal{O}_X) = 0$.

Proof. In view of the above theorem, it is enough to prove that (i) $\Leftrightarrow$ (ii). This we do. Assume that X is locally Gorenstein and consider a coresolution

$$0 \to \mathcal{O}_X \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \to \mathcal{I}^{d-1} \to \mathcal{L} \to 0$$

of X in which $\mathcal{I}^n$, $0 \leqslant n \leqslant d-1$ is injective in $\mathfrak{Qco}(X)$. Cover X with open affines $U_i = \operatorname{Spec}(A_i)$. For all $0 \leqslant n \leqslant d-1$, $\mathcal{I}^n|_{U_i}$ is an injective object in $\mathfrak{Qco}(U_i)$. Since X is a noetherian locally Gorenstein scheme of dimension $d < \infty$, A_i is Gorenstein of injective dimension less than or equal to d. Assume that $A_i \to I^\bullet$ is an injective coresolution of A_i of length d. Taking $\sim$ gives an injective coresolution $\widetilde{A}_i \to \widetilde{I}^\bullet$ of $\widetilde{A}_i$ of length d. This implies that $\mathcal{L}|_{U_i}$ is injective in $\mathfrak{Qco}(U_i)$. So $\mathcal{L}$ is locally injective. Hence $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{O}_X) < \infty$.

Now conversely, assume that $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{O}_X)$ is finite. Consider a cover of X by open affine subsets $U_i = \operatorname{Spec} A_i$, where each A_i is a noetherian ring. To prove (i), we have to show that each A_i is Gorenstein. Fix i and consider $A_i = A$. Let I be an arbitrary ideal of A. Let $x \in U = U_i$. Since $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{O}_X) < \infty$, there exists a cover of X by open subsets V_j such that $\mathcal{O}_X|_{V_j}$ is of finite injective dimension. Let V_j be such that $x \in V_j$. Assume that injective dimension of $\mathcal{O}_X|_{V_j}$ is less than or equal to d. Hence $\mathcal{E}xt^{d+1}_{\mathfrak{Q}(V_j)}(\widetilde{I}|_{V_j}, \widetilde{A}|_{V_j}) = 0$. This implies that $\mathcal{E}xt^{d+1}_{\mathfrak{Q}(U)}(\widetilde{I}, \widetilde{A})_x = 0$. So $\operatorname{Ext}^{d+1}_{A_x}(I_x, A_x) = 0$. Since all the ideals of A_x are of the form I_x, we deduce that $\operatorname{id}(A_x) < \infty$. So A_x is a Gorenstein local ring. But $x \in U$ was an arbitrary element, so A is Gorenstein. This completes the proof. $\square$

The proof of (ii) $\Rightarrow$ (i) of the above theorem shows that the stalk $\mathcal{I}_x$ of any quasi-coherent injective $\mathcal{O}_X$-module over a noetherian scheme of finite dimension is an injective $\mathcal{O}_x$-module.

Corollary 5.2.6. *Assume that X is a noetherian scheme of finite dimension. Then X is locally Gorenstein if and only if* $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{F}) < \infty$ *for any locally free sheaf $\mathcal{F}$ over X.*

Proof. The 'if' part follows from the above theorem, because $\mathcal{O}_X$ is a free module over itself. So assume that X is locally Gorenstein and consider a locally free sheaf $\mathcal{F}$ on X. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective coresolution of $\mathcal{F}$ in $\mathfrak{Qco}(X)$ and let $\mathcal{L}$ be its dth cosyzygy, where $d = \dim(X)$. Cover X with open affines $U_i = \operatorname{Spec} A_i$. So A_i is Gorenstein ring and $\widetilde{A}_i = \mathcal{O}_X|_{U_i}$. We show that $\mathcal{L}$ is locally injective. To this end, consider an open cover V_j of X such that $\mathcal{F}|_{V_j} = \bigoplus \mathcal{O}_X|_{V_j}$. Now $\bigoplus \mathcal{O}_X|_{U_i} = \bigoplus \widetilde{A}_i$ is of finite injective dimension at most d. Hence $\bigoplus \mathcal{O}_X|_{U_i \cap V_j}$ also has a coresolution by injective sheaves of finite length. So $\mathcal{L}|_{U_i \cap V_j}$ is injective $\mathcal{O}_X|_{U_i \cap V_j}$-module. Since $U_i \cap V_j$ provide an open cover of X, $\mathcal{L}$ is locally injective. Hence $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{F}) < \infty$. $\square$

A scheme X is called locally Cohen–Macaulay if its local rings all are Cohen–Macaulay local rings.

Corollary 5.2.7. *Let X be a noetherian scheme and $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module with* $\operatorname{Supp} \mathcal{F} = X$. *If* $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{F}) < \infty$*, then X is locally Cohen–Macaulay.*

Proof. (ii) $\Rightarrow$ (i). Since $\operatorname{lid}_{\mathfrak{Q}(X)}(\mathcal{F}) < \infty$, for any $x \in X$, $\operatorname{id}_{\mathcal{O}_{X,x}} \mathcal{F}_x < \infty$. Hence, by [Mat, Theorem 18.9], $\mathcal{O}_{X,x}$ is a Cohen–Macaulay local ring. So X is locally Cohen–Macaulay. $\square$

In the following, we aim to show that over noetherian Gorenstein schemes, complete cohomology modules can be computed using certain complete coresolutions of objects in the second variable. It is, in fact, an analogue of the Vogel's result over commutative noetherian Gorenstein rings, see [Mart, 2.1]. We preface the theorem with a definition.

Definition 5.2.8. Let $\mathcal{F} \in \mathfrak{Qco}(X)$. By a complete coresolution of $\mathcal{F}$ we mean an exact sequence of locally injective sheaves which agrees with an injective coresolution of $\mathcal{F}$ in degrees small enough.

Theorem 5.2.9. *Let X be a noetherian locally Gorenstein scheme of dimension d. Assume that $\mathcal{F}$ is an $\mathcal{O}_X$-module admitting a resolution by locally free sheaves. Then $\mathcal{F}$ has a complete coresolution.*

Proof. Let $\cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{F} \to 0$ be a resolution of $\mathcal{F}$ by locally free $\mathcal{O}_X$-modules and let, for any integer n, $\mathcal{F}_n \to \mathcal{I}_n^\bullet$ be an injective coresolution of $\mathcal{F}_n$ in $\mathfrak{Qco}(X)$. Moreover, consider an injective coresolution $\mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$. Putting these together, using the same argument as in the proof of Theorem 3.2.3, we get an exact sequence

$$\cdots \to \mathcal{L}_n \to \cdots \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{I}^d \to \mathcal{I}^{d+1} \to \cdots,$$

in which $\mathcal{L}_n$, for any integer n, is the dth cosyzygy of $\mathcal{F}_n$. By Corollary 5.2.6, $\mathcal{L}_n$, for any integer n, is locally injective. Hence the above resolution is a complete coresolution of $\mathcal{F}$. □

Lemma 5.2.10. *Let X be a noetherian locally Gorenstein scheme of dimension d. Then for any quasi-coherent sheaf $\mathcal{F}$ and any quasi-coherent injective sheaf $\mathcal{I}$, $\mathcal{E}xt^n_{\mathfrak{Q}(X)}(\mathcal{I}, \mathcal{F}) = 0$, for all $n > d$.*

Proof. Cover X with open affine subsets $U = \operatorname{Spec}(A)$, where A is a commutative noetherian ring. Since X is locally Gorenstein, A is a Gorenstein ring of dimension less than or equal to d. We show that $\mathcal{E}xt^n_{\mathfrak{Q}(X)}(\mathcal{I}, \mathcal{F})|_U = \mathcal{E}xt^n_{\mathfrak{Q}(U)}(\mathcal{I}|_U, \mathcal{F}|_U)$ vanishes for all $n > d$.

Since $\mathcal{I}|_U$ is an injective module over $\mathcal{O}_U = \widetilde{A}$, it is of the form $\widetilde{I}$, where I is an injective A-module. But the Gorensteinness of A implies that projective dimension of I is finite. Therefore, $\widetilde{I}$ is of homological dimension less than or equal to d. So $\mathcal{E}xt^n_{\mathfrak{Q}(U)}(\mathcal{I}|_U, \mathcal{F}|_U) = 0$, for all $n > d$. This implies the result. □

Theorem 5.2.11. *Let X be a noetherian Gorenstein scheme of dimension d and $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules. Let $\mathcal{L}^\bullet$ be a complete coresolution of $\mathcal{G}$. Then for any integer n,*

$$\widetilde{\mathcal{E}xt}^n_{\mathfrak{Q}(X)}(\mathcal{F}, \mathcal{G}) \cong \mathcal{H}_{-n}\mathcal{H}om\big(\mathcal{F}, \mathcal{L}^\bullet\big).$$

Proof. Consider a complete coresolution

$$\mathcal{L}^\bullet\colon \quad \cdots \to \mathcal{L}_1 \xrightarrow{\partial_1} \mathcal{L}_0 \xrightarrow{\partial_0} \mathcal{L}_{-1} \to \cdots$$

of locally injective $\mathcal{O}_X$-modules of $\mathcal{G}$. For an arbitrary integer n, set $\mathcal{K} := \operatorname{Ker} \partial_n$. In view of Lemma 5.2.2, the resolution $0 \to \mathcal{K} \to \mathcal{L}_n \to \mathcal{L}_{n-1} \to \mathcal{L}_{n-2} \to \cdots$ can be used to compute $\mathcal{E}xt^*_{\mathfrak{Q}(X)}(\ , \mathcal{K})$. On the other hand, by the above lemma, for all $j > d$ and for any quasi-coherent injective sheaf $\mathcal{I}$, $\mathcal{E}xt^j_{\mathfrak{Q}(X)}(\mathcal{I}, \mathcal{K}) = \mathcal{H}_{-j}\mathcal{H}om(\mathcal{I}, \mathcal{L}^\bullet_{\leqslant n}) = 0$. Since n is an arbitrary integer, this implies that $\mathcal{L}^\bullet$ is $\mathcal{H}om(\mathcal{I},\ )$-exact. Now consider an injective coresolution of $\mathcal{F}$ by quasi-coherent injective sheaves,

$$0 \to \mathcal{F} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to \cdots.$$

The exactness of the resolution $\mathcal{L}^\bullet$ under the functor $\mathcal{H}om(\mathcal{E}^j,\ )$ in view of the definition of $\overline{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K})$, implies easily that $\overline{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K}) = 0$, for all $i > 0$. So $\widetilde{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K}) \cong \mathcal{E}xt^i_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K})$, for all $i > 0$ and all sheaves of the form $\mathcal{K} := \operatorname{Ker}\partial_n$. Now fix $n \in \mathbb{Z}$ and let $p \leqslant n$ be such that $\mathcal{L}_i$ is equal to $\mathcal{I}_i$, for $i < p$, where $\mathcal{G} \to \mathcal{I}^\bullet$ is an injective coresolution of $\mathcal{G}$. Consider the sequences

$$0 \to \mathcal{G} \to \mathcal{I}_0 \to \mathcal{I}_{-1} \to \cdots \to \mathcal{I}_{p-1} \to \mathcal{L} \to 0$$

and

$$0 \to \mathcal{K} \to \mathcal{L}_n \to \mathcal{L}_{n-1} \to \mathcal{L}_{n-1} \to \cdots \to \mathcal{L}_{p-1} \to \mathcal{L} \to 0.$$

From the first resolution it follows, by splitting it to short exact sequences, that $\widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{G}) \cong \widetilde{\mathcal{E}\mathrm{xt}}^{n-p}_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{L})$. From the second resolution we get $\widetilde{\mathcal{E}\mathrm{xt}}^{n-p}(\mathcal{F},\mathcal{L}) \cong \widetilde{\mathcal{E}\mathrm{xt}}^1(\mathcal{F},\mathcal{K})$. But

$$\widetilde{\mathcal{E}\mathrm{xt}}^1_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K}) \cong \mathcal{E}xt^1_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K}) \cong \mathcal{H}_{-1}\mathcal{H}om\big(\mathcal{F},\mathcal{L}^\bullet_{\leqslant n}\big) \cong \mathcal{H}_{-n}\mathcal{H}om\big(\mathcal{F},\mathcal{L}^\bullet\big).$$

The result hence follows. □

In our next result, we compare Tate cohomology and complete cohomology sheaves over locally Gorenstein schemes.

Theorem 5.2.12. *Let X be a noetherian locally Gorenstein scheme of finite dimension. Then for any sheaves $\mathcal{F}$ and $\mathcal{K}$ that admit locally free resolutions,*

$$\widetilde{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K}) \cong \widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K}),$$

for all integers i. Conversely, if X is locally noetherian scheme such that for any coherent sheaves $\mathcal{F}$ of finite Gorenstein homological dimension, and any coherent sheaf $\mathcal{K}$ that admits a locally free resolution, $\widetilde{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{K}) \cong \widehat{\mathcal{E}xt}^i(\mathcal{F},\mathcal{K})$, then X is locally Gorenstein.

Proof. By Lemma 5.2.2, locally injective sheaves are in $\mathfrak{Coh}(X)^\perp$. So a complete coresolution of $\mathcal{K}$ is in fact a $\mathfrak{Coh}(X)^\perp$-complete coresolution. The result now follows from the above theorem and Theorem 3.2.8. For the converse note that $\mathcal{O}_X$ satisfies the conditions and so $\widetilde{\mathcal{E}\mathrm{xt}}^i_{\mathfrak{Q}(X)}(\mathcal{O}_X,\mathcal{O}_X) \cong \widehat{\mathcal{E}xt}^i(\mathcal{O}_X,\mathcal{O}_X)$. But $\widehat{\mathcal{E}xt}^i(\mathcal{O}_X,\mathcal{O}_X) = 0$. So, by Theorem 5.2.5, $\operatorname{lid}_{\mathcal{O}_X}\mathcal{O}_X < \infty$. Hence X is locally Gorenstein. □

Corollary 5.2.13. *Let X be a noetherian locally Gorenstein scheme of finite dimension and $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves. Then, for all integers n, $\widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{G})$ is coherent. Conversely, if $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G})$ is coherent, for some integer n and all coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, then X is locally Gorenstein.*

Proof. It is easy to see that, for any open subset U of X, $\widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(X)}(\mathcal{F},\mathcal{G})|_U \cong \widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(U)}(\mathcal{F}|_U,\mathcal{G}|_U)$. Cover X with open affines $U_i = \mathrm{Spec}(A_i)$. So, for any integer i, $\mathcal{F}|_{U_i}$ and $\mathcal{G}|_{U_i}$ admit locally free resolutions. Hence by the above theorem,

$$\widetilde{\mathcal{E}\mathrm{xt}}^n_{\mathfrak{Q}(U_i)}(\mathcal{F}|_{U_i},\mathcal{G}|_{U_i}) \cong \widehat{\mathcal{E}xt}^n(\mathcal{F}|_{U_i},\mathcal{G}|_{U_i}).$$

But it follows from the construction of the latter sheaf that, it is coherent. This implies the proof of the first part. For the converse, assume that $\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{G})$ is coherent, for some integer n and some coherent sheaves $\mathcal{F}$ and $\mathcal{G}$. Again consider a covering of X by open affine subsets $U = \mathrm{Spec}(A)$. For any $x \in X$, consider the coherent sheaf $\widetilde{A_x/xA_x}$ on U. By [H, II.Ex. 5.15], there is a coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}|_U \cong \widetilde{A_x/xA_x}$. By assumption

$$\widetilde{\mathcal{E}\mathrm{xt}}^n(\mathcal{F},\mathcal{F})|_U \cong \widetilde{\mathcal{E}\mathrm{xt}}^n(\widetilde{A_x/xA_x},\widetilde{A_x/xA_x}) \cong \widetilde{\mathrm{ext}}^n(A_x/xA_x, A_x/xA_x)^{\sim}$$

is coherent. So $\widetilde{\mathrm{ext}}^n(A_x/xA_x, A_x/xA_x)$ is finitely generated. Hence, by Theorem 5.1.6, A_x is Gorenstein local ring. Therefore X is locally Gorenstein. □

Acknowledgments

We would like to thank Robin Hartshorne and David A. Cox for reading the first draft of the manuscript and for their useful comments and hints on the paper that improved our exposition.

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